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Statistics on a Sphere

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Summary

Expectations deduced from the probability density functions of Fisher are used to develop further the statistics of points on a sphere. The paper presents unbiased estimators of the precision parameter κ in terms of vector deviations both for cases when the true direction $\boldsymbol{\mu}$ is known and unknown. On the basis of a one way random effect vector model, the scatter of various sampling distributions of means are derived as functions of the within (κ_w) and between (κ_b) sites scatter. The relations take the curvature of the sphere into account and extend the analysis of dispersion on a sphere to include highly-scattered distributions of palaeomagnetic data. In addition to the mean square method, which is modified by a new expression for the expectation of the mean square between sites, two alternative ways of estimating κ_w and κ_b are described. The latter statistics contribute to determine the confidence circle of the overall mean direction with unit weight to samples and sites respectively. Finally, the theory is applied to palaeomagnetic results from the Kaoko lavas of South-West Africa.

1. Introduction

The theory of errors developed by Gauss for linear random variables, was extended by Fisher (1953) to measurements of positions on a sphere. In this system, extensively used in palaeomagnetism, a population of points on a unit sphere is assumed to conform to a theoretical distribution, uniquely defined by a precision parameter κ and a true direction symbolized by the unit vector $\boldsymbol{\mu}$. Fisher derived the required statistics to estimate these parameters, the latter by using a mean direction and a confidence circle A_{95}.

Watson (1956) and Watson & Irving (1957) presented a series of statistical tests related to Fisherian statistics. In connection with the analysis of dispersion of remanent magnetization in rocks these include an F-test to test the significance of the between-sites scatter κ_b. Geophysicists take an interest in this dispersion firstly because it may be caused by secular variation of the Earth's magnetic field. Secondly it is of importance in estimating errors of means in connection with studies of the Earth's magnetic field throughout geological time (Nagata 1962; Irving 1964; McElhinny 1973). A problem arises, however, when quantitative estimates are to be made of κ_b, because the available method is approximate and claimed to be valid only for high precision parameters.

This paper concerns sampling theory and estimation theory related to Fisherian statistics and generalizes the analysis of dispersion on a sphere. It begins with deductions from the basic functions presented in Section 2 with the purpose of obtaining functional relations valid for the whole range of precision parameters. The vector error ω_i defined as the vector difference between a random unit vector $\mathbf{p}_i$ in a Fisherian distribution and the true mean value $\boldsymbol{\mu}$, provides an alternative way of referring to positions on a sphere. Section 3 concerns the expectations of the Cartesian components of ω_i (Z-axis along $\boldsymbol{\mu}$) in addition to the expectation $E(\omega_i^2)$ of the square of its magnitude. Knowing $\boldsymbol{\mu}$ the latter shows that 2 over the mean square of the vector errors forms an unbiased estimate of κ. Further when applied to a vector model $\mathbf{p}_{ij} = \boldsymbol{\mu} + \beta_j + \omega_{ij}$ (see equation (37)), which corresponds to the linear one way random effect model used for Gaussian analysis of variance (Hays 1972), the expectations lead to equation (39). It expresses the overall dispersion of sample means κ_a as a function of the within-site scatter κ_w and between-sites scatter κ_b. This formula and other expressions in Section 5, derived by taking the curvature of the sphere into account, reduce, for high precision parameters, to related equations commonly used in palaeomagnetism.

Section 4 deals with the expectation of R, where R symbolizes the scalar value of the vector sum $\mathbf{R}$ of a sample of N random unit vectors. The validity of the approximate equation (29) is discussed and made use of in Section 3 as well as in Sections 5 and 6. In the first place it results in an unbiased estimator of κ for the frequently occurring events when $\boldsymbol{\mu}$ is unknown. Then, for the general case when a between-sites scatter does exist, it contributes to derive the overall dispersion of site means κ_l by means of κ_w and κ_b, in addition to similar functional relations for the scatter κ_{ma} and κ_{ml} of means of sample means and means of site means respectively (equations (45) and (47)). The error or the confidence circle A_{95} of the overall mean value with unit weight on samples and sites respectively are subsequently estimated by means of κ_{ma} and κ_{ml} (equations (4) or (6)). Section 6 reviews the F-test to judge the significance of a possible between-sites scatter κ_b and presents a new formula (60) for the expectation of the mean square between sites which make the analysis of dispersion less dependent on the magnitudes of the precision parameters. This method of estimating κ_w and κ_b is referred to as the 'Mean square method' to distinguish it from the 'Circular standard deviation method' and the 'Overall dispersion method' described in Section 7. In Section 8 the new set of formulae is applied to dispersion analysis of the remanent magnetization distribution observed in the Kaoko lavas from South-West Africa (Gidskehaug, Creer & Mitchell 1975).

2. Basic functions

Any direction in space may be represented either by a unit vector $\mathbf{p}$ from the origin 0 of a XYZ-coordinate system or by the point of intersection q between $\mathbf{p}$ and a unit sphere centred about 0. Generally $q(\theta, \phi)$ will be referred to in a spherical coordinate system with the polar axis along the positive Z-axis.

In this frame equation (1) defines a Fisherian distribution of points on a unit sphere.

$$f_1(\theta, \phi) = (\kappa/4\,\pi \sinh \kappa)\, e^{\kappa \cos\theta} \tag{1}$$

$f_1(\theta, \phi)$ symbolizes the probability density of a random point $q(\theta, \phi)$, specified by its angular distance θ from the true direction $\boldsymbol{\mu}$ and by the azimuthal angle ϕ. Equation (1) shows that a Fisherian distribution is symmetrical about $\boldsymbol{\mu}$ and a high value of the precision parameter κ corresponds to a small scatter (Fisher 1953).

Equation (2) expresses the probability dP of observing a point in the infinitesimal area dA.

$$dP = f_1(\theta, \phi)\quad dA = (\kappa/4\,\pi \sinh \kappa)\, e^{\kappa \cos\theta} \sin\theta\, d\theta\, d\phi. \tag{2}$$

It concerns the simultaneous distribution of θ and ϕ and results in the unconditional or marginal distribution of θ by an integration from $\phi = 0$ to $\phi = 2\pi$. A subsequent substitution of $c = \cos\theta$, provides the probability density function of the marginal distribution of c.

$$f_3(c) = \frac{\kappa e^{\kappa c}}{2 \sinh \kappa}. \tag{3}$$

Directions or points on the unit sphere having a constant value of θ will define a circle about the true mean direction. The 'radius' θ_p of the circle within which a random direction falls with probability $P = P(\theta < \theta_p)$ is derived from either equation (2) or (3)

$$\theta_p = \arccos\left(\frac{1}{\kappa} \ln(e^{\kappa} - P(e^{\kappa} - e^{-\kappa}))\right). \tag{4}$$

A value of P equal to 0·63, for example, results in the circular standard deviation θ_{63}, which frequently serves as an alternative expression of dispersion. For large value of κ the approximate formulae (5) and (6) apply at the 0·63 and 0·95 probability level respectively

$$\theta_{63} \simeq 81\kappa^{-\frac{1}{2}} \text{ degrees} \tag{5}$$

$$\theta_{95} \simeq 140\kappa^{-\frac{1}{2}} \text{ degrees} \tag{6}$$

θ_{63} calculated by equation (5) and by the general formula (4) agree to better than one tenth of a degree when $\kappa > 12$. However, to get the same accuracy by equation (6) κ has to exceed 60 (Gidskehaug 1968, p. 62).

Now let $\mathbf{R}$ be the vector sum of a sample of N unit vectors randomly selected from a Fisherian population and let R symbolize the length of $\mathbf{R}$. Fisher (1953) deduced the marginal distribution of R and the conditional distribution of C on given R, in terms of the probability density functions (7) and (8) respectively.

$$f_7(R) = \left(\frac{\kappa}{2 \sinh \kappa}\right)^N \frac{2 \sinh(\kappa R)}{\kappa}\, \phi_N(R) \tag{7}$$

$$f_8(C/R) = \frac{\kappa R\, e^{\kappa R C}}{2 \sinh(\kappa R)} \tag{8}$$

C symbolizes cosine of the angular deviation between $\mathbf{R}$ and the true direction and $\phi_N(R)$ refers to the following function of N and R

$$\phi_N(R) = \frac{1}{(N-2)!}\left\{(N-R)^{N-2} - N(N-R-2)^{N-2} + \ldots + (-1)^s \frac{N!}{s!(N-s)!}(N-R-2s)^{N-2}\right\}. \tag{9}$$

In equation (9) s is the largest integer less than $\frac{1}{2}(N-R)$. Thus the density function of R, which is defined between 0 and N, has 'discontinuity' points at $N-2$, $N-4$, $N-6$ .... For values of R larger than $N-2$ ($s = 0$) only the first term in equation (9) appears and equation (7) reduces to

$$f_{10}(R) = \frac{1}{(2 \sinh \kappa)^N} \frac{(N-R)^{N-2}}{(N-2)!} \kappa^{N-1} (e^{\kappa R} - e^{-\kappa R}). \tag{10}$$

Fisher (1953) showed that the direction of **R** constitutes the best point estimate of **μ**. Further, on the assumption that κ is large enough to ignore $e^{-\kappa}$ ($\kappa > 3$), he deduced from equation (8) and (10) the uncertainty of the observed mean direction or the radius A_{95} of the 95 per cent circle of confidence

$$A_{95} = \arccos\left(1-\frac{N-R}{R}(20^{1/(N-1)}-1)\right) \tag{11}$$

and an estimate of the precision parameter κ given by

$$k = \frac{N-1}{N-R}. \tag{12}$$

Finally, for large κ ($\kappa > 3$) Watson (1956) deduced from equation (10) that the statistic $2\kappa(N-R)$ has a Chi-square distribution with $2(N-1)$ degrees of freedom. This useful information will be symbolized by

$$\chi^2_{2(N-1)} = 2\kappa(N-R). \tag{13}$$

3. Functional relations derived from cosine of the angular deviation of a random unit vector from the true direction

By means of equation (3) the expectation of $c = \cos\theta$ is found to be equal to Langevin's function of κ.

$$E(c) = \int_{-1}^{+1} cf_3(c)\,dc = \coth\kappa - \frac{1}{\kappa} = L(\kappa) \tag{14}$$

which means that the expectation of c approaches unity as κ increases.

Let the unit vector $\mathbf{p}_i$ represent a random observation in a Fisherian population (note that p_x is short for p_{ix})

$$\mathbf{p}_i = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}. \tag{15}$$

Then equation (14) also expresses the expectation of **p**'s component along the true direction $\boldsymbol{\mu} = \mathbf{k}$, because $p_z = \cos\theta$. Further let the vector differences between $\mathbf{p}_i$ and **μ** be referred to as the 'vector error' or the 'vector deviation' $\boldsymbol{\omega}_i$ and thus

$$\mathbf{p}_i = \boldsymbol{\mu} + \boldsymbol{\omega}_i. \tag{16}$$

Since the distribution is symmetrical about **μ** (i.e. the Z-axis) the expectation of the X and Y components of $\boldsymbol{\omega}_i$ both become zero.

$$E(\omega_x) = E(\omega_y) = 0 \tag{17}$$

while

$$E(\omega_z) = E(p_z - 1) = \coth\kappa - \frac{1}{\kappa} - 1 \approx -\frac{1}{\kappa}. \tag{18}$$

The approximate expression $E(\omega_z) = -1/\kappa$ is accurate to better than 0·001 when $\kappa = 4$ and improves still further as κ increases. The dot product of the vector error by itself or the square of its scalar value reduces to $\omega_i^2 = 2(1-p_z) = -2\omega_z$. Consequently equation (19) provides an alternative way to estimate the precision parameter which is independent of the orientation of the co-ordinate system.

$$E(\omega_i^2) = 2\left(1-\coth\kappa + \frac{1}{\kappa}\right) \approx \frac{2}{\kappa}. \tag{19}$$

Let $\mathbf{r} = \mathbf{R}/R$ be the unit vector of the vector sum of a sample of N random observations $\mathbf{p}_i$ ($i = 1, \ldots N$). A comparison of equations (3) and (8) show that for a given value of R, the mean direction **r** conforms to a Fisherian distribution with a precision parameter equal to κR. Thus by writing

$$\mathbf{r} = \boldsymbol{\mu} + \boldsymbol{\varepsilon} \tag{20}$$

we get in a similar way as above

$$E(\varepsilon^2) = 2\left(1-\coth R\kappa + \frac{1}{R\kappa}\right) \approx \frac{2}{R\kappa}. \tag{21}$$

Suppose **μ** is known and that a sample of N random unit vectors is taken from a Fisherian distribution. Then on the basis of equation (19)

$$2/k = \overline{\omega^2} = \frac{\sum^N \omega_i^2}{N} = \frac{2(N-R_z)}{N} = \frac{2(N-R\cos v)}{N} \tag{22}$$

where v is the angle between **R** and **μ** (z-axis),
i.e.

$$k = 2/\overline{\omega^2} = \frac{N}{N-R\cos v}. \tag{23}$$

Consequently, twice the reciprocal mean square of the vector error from the true mean is an unbiased estimator of the precision parameter κ. This estimator corresponds to equation (1·15) given by Watson (1956).

Generally, however, **μ** is unknown. By writting $\mathbf{p}_i = \mathbf{r} + \boldsymbol{\delta}_i$ and fixing the z-axis along **r** (which symbolizes the unit vector of **R**), then again from equation (19)

$$2/k_r = \overline{\delta^2} = \frac{\sum^N \delta_i^2}{N} = \frac{2(N-R)}{N}, \text{ so } k_r = \frac{N}{N-R}. \tag{24}$$

Equation (29) in the next section shows that $1/k_r$ is a biased estimator of $1/\kappa$. However, in terms of the vector errors an unbiased estimator of $1/\kappa$ is given by

$$1/k = \frac{\sum^N \delta_i^2}{2(N-1)} = \frac{N-R}{N-1} \tag{25}$$

which applies for the whole range of $\kappa > 3$ and for N values at least up to 3κ. Note that equation (25) corresponds to equation (12).

4. The expectation of R

In this section $P_{2,0}$, $P_{4,2}$ and $P_{4,0}$ refer to the probability of observing R in the intervals $(N-2, N)$, $(N-4, N-2)$ and $(N-4, N-0)$ respectively, and similar suffixes will be employed to distinguish between the expectation of R derived over the same intervals. Preferably the expectation of R should be determined by integration over the whole range of possible values from zero to N. However, due to the complexity of the general density function (7) the interval $(N-2, N)$ will be considered first. Equation (10) defines the probability density function of R in this domain. Evidently it may be used to deduce the expectation of R if the distribution of R confines to the interval, that is if the probability $P(R > N-2)$ of observing a value of R larger than $N-2$ lies close to one.

$$P_{2,0} = P(R > N-2) = \int_{N-2}^{N} f_{10}(R)\,dR$$

$$= \frac{1}{(2\sinh\kappa)^N}\left(e^{\kappa N} + (-1)^N e^{-\kappa N} - e^{\kappa(N-2)}\sum_{n=0}^{N-2}\frac{(2\kappa)^n}{n!}\right.$$

$$\left. - e^{-\kappa(N-2)}\sum_{n=0}^{N-2}(-1)^{n+N}\frac{(2\kappa)^n}{n!}\right). \tag{26}$$

The two sums in equation (26) contain the first $(N-1)$ terms in the power series of $e^{2\kappa}$ and $e^{-2\kappa}$ respectively. This means that for a given value of κ the probability decreases and approaches zero as N increases. Consequently the probability $P(R > N-2)$ does not depend on the absolute value of κ, but on how large κ is relative to N. For values of κ big enough to ignore the '$e^{-\kappa}$ terms' in equation (10) equation (26) reduces to

$$P_{2,0} = 1 - e^{-2\kappa}\sum_{n=0}^{N-2}\frac{(2\kappa)^n}{n!} \tag{27}$$

and the expectation of R becomes

$$E(R)_{2,0} = \int_{N-2}^{N} R f_{10}(R)\,dR$$

$$= \left\{\frac{e^{\kappa R}}{e^{\kappa N}}\sum_{n=0}^{N-2}\left(R - \frac{N-1-n}{\kappa}\right)\frac{\kappa^n(N-R)^n}{n!}\right\}_{N-2}^{N}$$

$$= N - \frac{N-1}{\kappa} - e^{-2\kappa}\sum_{n=0}^{N-2}\left(N-2-\frac{N-1-n}{\kappa}\right)\frac{(2\kappa)^n}{n!} \tag{28}$$

which if $N \lesssim \kappa$ (see Fig. 1) reduces to

$$E(R) = N - \frac{N-1}{\kappa} = (1-1/\kappa)N + 1/\kappa. \tag{29}$$

$E(R)$ defined by equation (29) is a linear function of N. Consequently for a fixed value of κ, $E(R)$ increases steadily as N increases, while due to the third term in equation (28), then $E(R)_{2,0}$ first reaches a maximum value, then quickly approaches zero for still higher values of N. Calculations on the basis of equations (27), (28) and (29) show that $P_{2,0}$ remains practically equal to 1, while $E(R)_{2,0} = E(R)$ to better than 0·001 if $N \leqslant \kappa$ (see Figs 1 and 2). Accordingly, we may preliminarily conclude that provided $N \leqslant \kappa$, equation (29) constitutes a good approximation of the expectation of R for the whole range of $\kappa \geqslant 3$.

The calculations are now extended to a larger domain. On the basis of equations (7) and (9), the probability of observing R in the interval $(N-4, N-2)$ is found to be, for $\kappa > 3$

$$P_{4,2} = P(N-4 < R < N-2)$$

$$= e^{-2\kappa}\sum_{n=0}^{N-2}\frac{(2\kappa)^n}{n!} - Ne^{-2\kappa} + e^{-4\kappa}\sum_{n=0}^{N-2}(N-2^n)\frac{(2\kappa)^n}{n!} \tag{30}$$

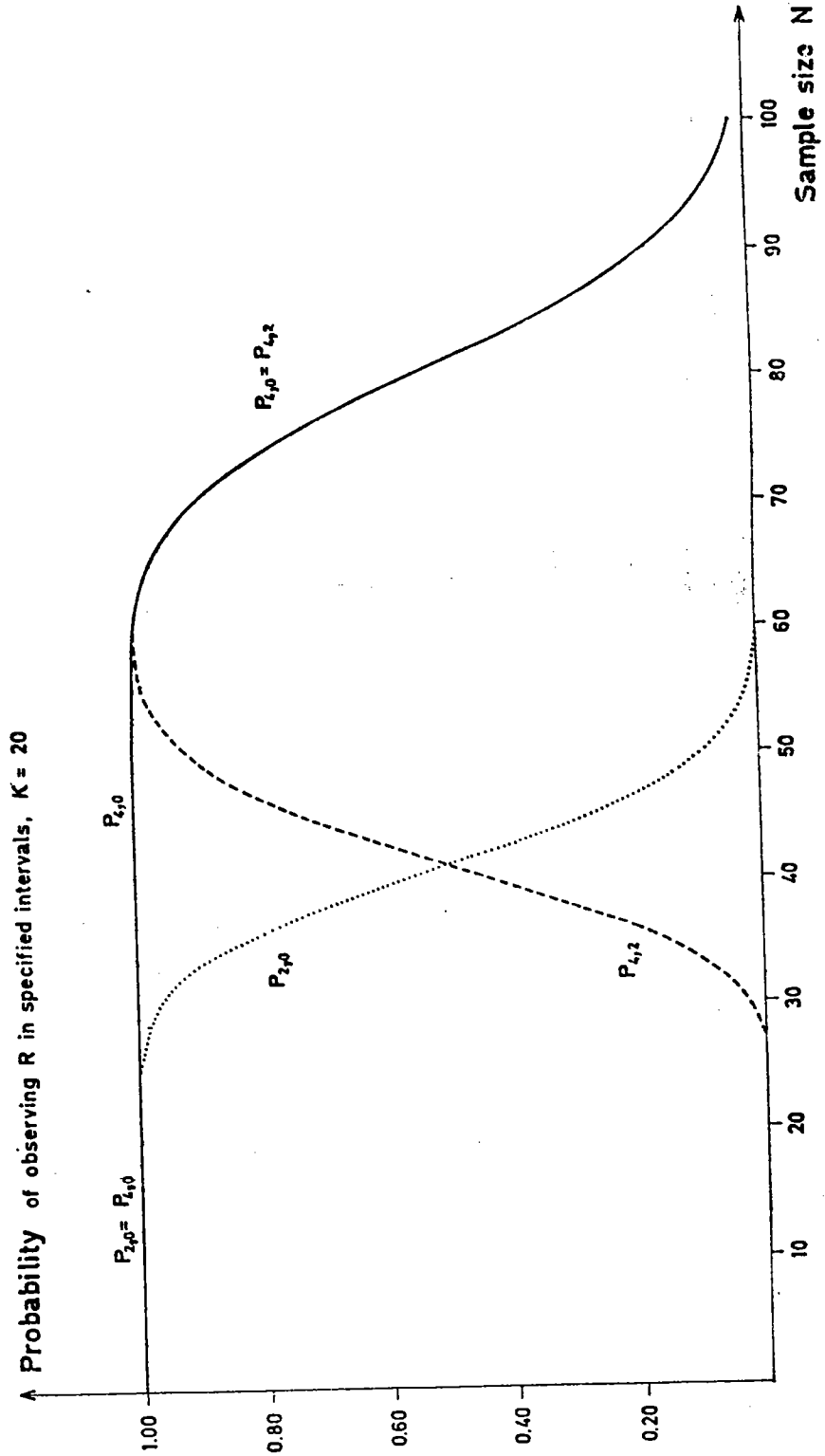

FIG. 1. The probability of observing R in restricted intervals graphically displayed as a function of the sample size N for a fixed value of the precision parameter κ. $P_{2,0}$, $P_{4,0}$ and $P_{4,2}$ symbolize the probability of observing R in the intervals $(N-2, N)$, $(N-4, N-2)$ and $(N-4, N)$ respectively.

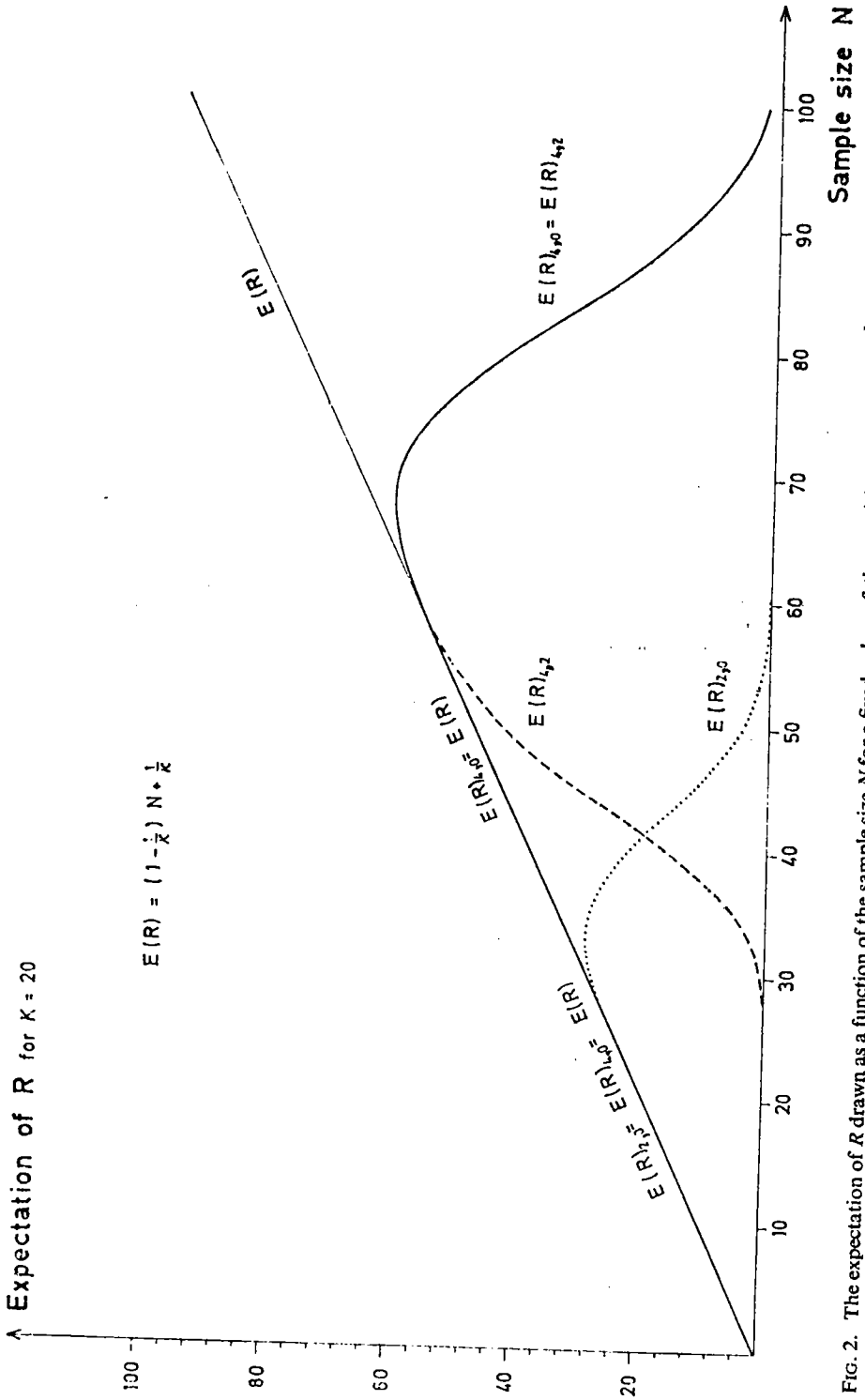

FIG. 2. The expectation of R drawn as a function of the sample size N for a fixed value of the precision parameter κ. $E(R)_{2,0}$, $E(R)_{4,0}$, $E(R)_{4,2}$ and $E(R)_{4,0}$ denote the expectation of R deduced by integration over the intervals specified in Fig. 1.

which together with equation (27) leads to

$$P_{4,0} = P(R > N-4) = 1 - Ne^{-2\kappa} + e^{-4\kappa} \sum_{n=0}^{N-2} (N-2^n) \frac{(2\kappa)^n}{n!}. \qquad (31)$$

In the same intervals the following formulae for the expectation of R emerges

$$E(R)_{4,2} = e^{-2\kappa} \sum_{n=0}^{N-2} \left(N-2-\frac{N-1-n}{\kappa}\right) \frac{(2\kappa)^n}{n!} - e^{-2\kappa} N\left(N-2-\frac{N-1}{\kappa}\right)$$

$$+ e^{-4\kappa} \sum_{n=0}^{N-2} (N-2^n)\left(N-4-\frac{N-1-n}{\kappa}\right) \frac{(2\kappa)^n}{n!} \qquad (32)$$

and

$$E(R)_{4,0} = N - \frac{N-1}{\kappa} - e^{-2\kappa} N\left(N-2-\frac{N-1}{\kappa}\right)$$

$$+ e^{-4\kappa} \sum_{n=0}^{N-2} (N-2^n)\left(N-4-\frac{N-1-n}{\kappa}\right) \frac{(2\kappa)^n}{n!}. \qquad (33)$$

The relations are illustrated in Fig. 1, which shows $P_{2,0}$, $P_{4,2}$ and $P_{4,0}$ as functions of N for a chosen precision parameter $\kappa = 20$. Notice that $P_{4,2} = 0$ when $N = \kappa$, then it increases to a maximum at about $N = 3\kappa$ and falls to zero for higher values of N. $E(R)_{4,2}$ behaves in a similar way in Fig. 2, where in addition $E(R)_{2,0}$, $E(R)_{4,0}$ and $E(R)$ are graphically displayed. Still $E(R)_{4,0} = E(R)$, provided the probably $P_{4,0}$ of finding a value of R larger than $N-4$ is close to one. Table 1, which lists $P_{4,0}$ in per cent for various values of κ and N suggests that $P_{4,0} \approx 100$ per cent if $N < 3\kappa$. Consequently by including the interval $(N-4, N-2)$ the validity of equation (29) has been greatly extended. Most likely it can be proved to apply for even higher values of N ($> 3\kappa$) by considering new intervals, and possibly constituting the general formula for the expectation of R obtaining by integrating R from zero to N.

5. The dispersion of sampling distributions of means deduced on the basis of a one way random effect vector model

The following description uses palaeomagnetic terminology (Irving 1964). To study the distribution of remanent magnetization in a rock formation such as a pile of lava flows, N_j (generally five or more) oriented pieces of rocks called samples are collected from the jth of B sites. $\mathbf{p}_{ij}$ will denote the mean direction of stable remanent magnetization directions measured in a number of specimens cut from the ith rock sample of site j. The intention of this section is to deduce the dispersion κ_a and κ_l of the overall distribution of sample means $\mathbf{p}_{ij}$ and site means $\mathbf{r}_j$ respectively as a function of the within and between sites scatters. These expressions then form the basis of similar formulae for the scatter κ_{ma} and κ_{ml} of the means of sample means **ma** and the means of site means **ml**.

Let $\mathbf{p}_{ij}$ be the ith observation randomly selected from the jth population in a set of Fisherian populations having equal precision parameters κ_w and true directions $\boldsymbol{\mu}_j$. Let also a randomly chosen $\boldsymbol{\mu}_j$ be Fisherian distributed with precision parameters κ_b about an overall true direction $\boldsymbol{\mu}$. In palaeomagnetism κ_w and κ_b correspond to the within and between-sites scatter respectively, provided $\boldsymbol{\mu}$ denotes the true remanent magnetization direction of the rock formation in question and $\boldsymbol{\mu}_j$ symbolizes the true direction

Table 1

The probability $P_{4,0} = P(R > N-4)$ of observing R in the interval $(N-4, N)$ for different precision parameters κ and sample sizes N.

N \ κ	5	10	15	20	25	30	35	40	45	50	55	60
5	99·98	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00
10	99·76											100·00
15	93·38											100·00
20	61·86	99·49										100·00
25	21·25	99·74										100·00
30	3·43	97·06										100·00
35	0·27	84·86	99·99									100·00
40	0·01	58·40	99·84									100·00
45	0·00	28·34	98·67									100·00
50		9·25	93·50	99·99								100·00
55		2·00	79·76	99·91								100·00
60		0·29	56·86	99·39								100·00
65		0·03	31·96	97·10								100·00
70		0·00	13·70	90·31	99·96							100·00
75			4·42	76·36	99·72							100·00
80			1·07	55·94	98·67							100·00
85			0·20	34·20	95·37	99·98						100·00
90			0·03	17·04	87·62	99·87						100·00
95			0·00	6·84	73·90	99·38						100·00
100				2·20	55·32	97·78	99·99					100·00
110				0·12	19·63	85·36	99·71					100·00
120				0·00	3·49	54·85	96·80	99·97				100·00
130					0·30	21·71	83·43	99·49				100·00
140					0·01	4·82	54·49	95·79	99·93			100·00
150	0·00	0·00	0·00	0·00	0·00	0·58	23·41	81·78	99·20	100·00	100·00	100·00

of the site j. From these assumptions it follows that an observation $\mathbf{p}_{ij}$ randomly selected from the rock formation has a Fisherian distribution about $\boldsymbol{\mu}$, with a precision parameter denoted by κ_a (Fisher 1953, Section 2·33).

Within the given frame the overall dispersion of sample means κ_a now may be derived as a function of κ_w and κ_b by means of the vector presentations and functional relations presented in Section 3. Suppose the unit vectors $\mathbf{p}_{ij}$, $\boldsymbol{\mu}_j$ and $\boldsymbol{\mu}$ originate from the origin of a Cartesian co-ordinate system chosen to make the Z-axis coincide with $\boldsymbol{\mu}$. Further let the vector differences $\boldsymbol{\omega}_{ij}$, $\boldsymbol{\beta}_j$ and $\boldsymbol{\alpha}_{ij}$ be defined by the equations

$$\mathbf{p}_{ij} = \boldsymbol{\mu}_j + \boldsymbol{\omega}_{ij} \text{ leading to } E(\omega^2{}_{ij}) = 2/\kappa_w \tag{34}$$

$$\boldsymbol{\mu}_j = \boldsymbol{\mu} + \boldsymbol{\beta}_j \text{ leading to } E(\beta_j{}^2) = 2/\kappa_b \tag{35}$$

$$\mathbf{p}_{ij} = \boldsymbol{\mu} + \boldsymbol{\alpha}_{ij} \text{ leading to } E(\alpha^2{}_{ij}) = 2/\kappa_a. \tag{36}$$

In analogy of Gaussian statistics β_j may be referred to as the between-sites *effect*. The equations to the right imply that the values of the different precision parameters have to exceed (about) three for the approximate formulae of equation (18) and (19) to be valid. The vector equations on the other hand indicate

$$\boldsymbol{\alpha}_{ij} = \boldsymbol{\omega}_{ij} + \boldsymbol{\beta}_j \tag{37}$$

or

$$\alpha^2{}_{ij} = \omega^2{}_{ij} + \beta_j{}^2 + 2(\omega_{ijx}\,\beta_{jx} + \omega_{ijy}\,\beta_{jy} + \omega_{ijz}\,\beta_{jz}). \tag{38}$$

If the components of ω_{ij} and β_j are independent of each other the expectation of equation (38) results in

$$\frac{1}{\kappa_a} = \frac{1}{\kappa_w}\,\frac{\kappa_b}{\kappa_b - 1} + \frac{1}{\kappa_b}. \tag{39}$$

In the above calculations $E(\beta_{jx}) = E(\beta_{jy}) = 0$ and the expectation of the Z-component of $\boldsymbol{\omega}_{ij}$ follow from equation (37) and (18). (Notice that $E(\omega_{ijz}) \neq -1/\kappa_w$ since the Z-axis does not coincide with $\boldsymbol{\mu}_j$.)

$$E(\omega_{ijz}) = E(\alpha_{ijz}) - E(\beta_{jz}) = -1/\kappa_a + 1/\kappa_b. \tag{40}$$

Next we consider the sampling distribution of site means. Let $\mathbf{r}_j$ denote the unit vector of the vector sum $\mathbf{R}_j$ of N_j random observations $\mathbf{p}_{ij}$ from site j. According to equation (21), $\mathbf{r}_j$ has a within-site scatter about $\boldsymbol{\mu}_j$ equal to $R_j\,\kappa_w$ and a overall distribution about the true direction $\boldsymbol{\mu}$ assumed to be Fisherian with precision parameter κ_j'. In a similar way to (39)

$$\frac{1}{\kappa_j'} = \frac{1}{R_j\,\kappa_w}\,\frac{\kappa_b}{\kappa_b - 1} + \frac{1}{\kappa_b}. \tag{41}$$

Replacement of R_j with its expectation obtained by equation (29) leads to

$$\frac{1}{\kappa_j} = \frac{1}{N_j(\kappa_w - 1) + 1}\,\frac{\kappa_b}{\kappa_b - 1} + \frac{1}{\kappa_b} \tag{42}$$

which expresses the dispersion of the conditional sampling distribution of site means given $R_j = E(R_j)$. Equation (39) and (42) show

$$\kappa_a < \kappa_j < \kappa_b. \tag{43}$$

We may proceed with the sampling distributions of means of sample means and

$N = \sum_{j=1}^{B} N_j$ observations from B sites

$$\mathbf{ma} = \sum_{j=1}^{B} \sum_{i=1}^{Nj} \mathbf{p}_{ij}/R_N = \mathbf{R}_N/R_N. \tag{44}$$

If a random observation $\mathbf{p}_{ij}$ belongs to a Fisherian population with precision parameter κ_a about the true direction $\boldsymbol{\mu}$, then **ma** will also be Fisherian distributed about $\boldsymbol{\mu}$ with an expected precision parameter

$$\kappa_{ma} = \kappa_a E(R_N) = N\kappa_a - N + 1 = \frac{N\kappa_w\kappa_b(\kappa_b-1)}{\kappa_b^2+\kappa_w(\kappa_b-1)} - N + 1. \tag{45}$$

Finally let **ml** symbolize the unit vector of the resultant $\mathbf{R}_B$ of B random site means $\mathbf{r}_j$.

$$\mathbf{ml} = \sum_{j=1}^{B} \mathbf{r}_j/R_B = \mathbf{R}_B/R_B. \tag{46}$$

Again if $\mathbf{r}_j$ originates from a population with precision parameter κ_l about $\boldsymbol{\mu}$, **ml** will belong to a Fisherian distribution with true direction $\boldsymbol{\mu}$ and an expected dispersion κ_{ml}

$$\kappa_{ml} = \kappa_l E(R_B) = B\kappa_l - B + 1 = \frac{B\kappa_b(\kappa_b-1)(N_j\kappa_w - N_j + 1)}{\kappa_b^2 + (\kappa_b-1)(N_j\kappa_w - N_j + 1)} - B + 1. \tag{47}$$

The relations derived above apply for the whole range of $\kappa > 3$, provided the condition discussed in association with equation (29) is fulfilled.

For comparison, it is of interest to see what the various expressions reduce to for high values of κ_w and κ_b. Thus, equations (39) and (42) become

$$1/\kappa_a = 1/\kappa_w + 1/\kappa_b \tag{48}$$

$$1/\kappa_l = 1/N_j\kappa_w + 1/\kappa_b. \tag{49}$$

Further, if κ_a and κ_l are large enough to assume that $E(R_N) = N$ and $E(R_B) = B$ equations (45) and (47) reduce to

$$1/\kappa_{ma} = 1/N\kappa_w + 1/N\kappa_b \tag{50}$$

$$1/\kappa_{ml} = 1/N\kappa_w + 1/B\kappa_b \tag{51}$$

of which the latter corresponds to equation (17) given by Watson & Irving (1957), to equation (4.23) of Irving (1964) or to equation (3.31) of McElhinny (1973).

6. The expectation of mean square within sites and mean square between sites

The significance of a possible between-sites effect should be tested prior to estimation of the within and between-sites scatter (Watson & Irving 1957). If the remanent magnetization distribution of each site in a rock formation conforms to a Fisherian distribution with equal precision parameter κ_w and if no between-sites scatter exists (i.e. $\kappa_b = \infty$ or $\beta_j^2 = 0$) then equation (39) reduces to $\kappa_a = \kappa_w$ and according to relation (13)

$$\chi^2_{2(N-1)} = 2\kappa_w(N - R_N). \tag{52}$$

By using the previous notations the expression on the right-hand side of equation (52) splits into the two following terms (which are Chi-square distributed as well)

$$\chi^2_{2(N-B)} = 2\kappa_w \sum_{j=1}^{B} (N_j - R_j) \tag{53}$$

$$\chi^2_{2(B-1)} = 2\kappa_w \left(\sum_{j=1}^{B} R_j - R_N \right). \tag{54}$$

Consequently, the ratio between the mean square within sites

$$\text{MS-within} = \frac{\sum^{B}(N_j - R_j)}{2(N-B)} \tag{55}$$

and the mean square between sites

$$\text{MS-between} = \frac{\sum^{B} R_j - R_N}{2(B-1)} \tag{56}$$

conforms to a F-distribution in case of a true null-hypothesis: 'No between site scatter exists'. On the other hand if the null-hypothesis has to be rejected, κ_w and κ_b may be estimated from the expectation of equation (55) and (56). If the number of samples N_j in each of the B sites is equal, then from the relations (see equation (29))

$$E(R_j) = N_j - \frac{N_j - 1}{\kappa_w} \tag{57}$$

$$E(R_N) = N - \frac{N-1}{\kappa_a} \tag{58}$$

we get

$$E(\text{MS-within}) = \frac{1}{2}\frac{1}{\kappa_w} \tag{59}$$

$$E(\text{MS-between}) = \frac{1}{2(B-1)} \left(\frac{N-1}{\kappa_b} - \frac{N-B}{\kappa_w} + \frac{(N-1)\kappa_b}{\kappa_w(\kappa_b-1)} \right). \tag{60}$$

The above 'Mean square' method to estimate κ_w and κ_b may apply for the whole range of precision parameters larger than three. However, the basic relation (52) for the F-test is derived from the density function of R in the interval $(N-2, N)$. Accordingly care must be taken when $P_{2,0} = P(R > N-2)$ becomes significantly less than one (see Section 4) and possibly the F-test breaks down for values of N larger than κ_a. Before leaving the 'Mean square' method it is of interest to notice, that the expectation of MS-between sites reduces to a relation similar to that given by Watson & Irving (1957), i.e.

$$E(\text{MS-between}) = \frac{1}{2} \left(\frac{1}{\kappa_w} + \frac{(N-1)/(B-1)}{\kappa_b} \right) \tag{61}$$

if equation (48) is used instead of (39) in the above calculations.

7. Alternative methods of estimating the between- and within-site scatter

If a between-sites effect exists, equations (39) and (42) may be used to estimate κ_w and κ_b on the basis of the overall dispersion of sample means and site means. An elimination of κ_w from the two equations and a subsequent division of the algebraic expression by $\kappa_b - 1$ results in the following equation of degree three

$$\kappa_b^3 + p\kappa_b^2 + q\kappa_b + s = 0 \tag{62}$$

where

$$p = \frac{(N_j-1)(1+\kappa_l+\kappa_a-\kappa_a\kappa_l)}{N_j\kappa_a-\kappa_l-N_j+1}$$

$$q = \frac{-(N_j-1)(\kappa_a+\kappa_l+\kappa_a\kappa_l)}{N_j\kappa_a-\kappa_l-N_j+1}$$

$$s = \frac{(N_j-1)\kappa_a\kappa_l}{N_j\kappa_a-\kappa_l-N_j+1}.$$

Solutions of equation (62) are readily obtained on a computer. The fact that only positive and real values of the precision parameters will be considered and the restrictions imposed by relation (43) are generally sufficient to eliminate two of the three possible solutions of κ_b. κ_w is then obtained by inserting the solution of κ_b in either equation (39) or (42).

Estimates of κ_a and κ_l used in the above method are determined by equation (12). Preferably the number of samples in each site should be equal, but if not, a mean value might be inserted in equation (62).

As already indicated by relation (43), κ_a and κ_l must obey certain rules. If κ_a is kept fixed and κ_l allowed to vary a discontinuity in $\kappa_w(\pm\infty)$ exists if $\kappa_l = \kappa_a$ and a discontinuity in $\kappa_b(\pm\infty)$ appears when $\kappa_l = N_j\kappa_a - N_j + 1$. This suggests that in order to obtain positive solutions of both κ_w and κ_b, κ_l must be confined to the interval

$$\kappa_a < \kappa_l < N_j\kappa_a - N_j + 1 \tag{63}$$

or for a given κ_l the possible values of κ_a must be confined to the interval

$$(\kappa_l + N_j - 1)/N_j < \kappa_a < \kappa_l. \tag{64}$$

In either case, one discontinuity corresponds to no within-site scatter and the other to no between-sites scatter. From the approximate formulae (48) and (49) explicit expressions of the within and between-site scatter may be obtained.

$$\kappa_{w\,app} = \frac{N_j-1}{N_j}\,\frac{\kappa_a\kappa_l}{(\kappa_l-\kappa_a)} \tag{65}$$

$$\kappa_{b\,app} = \frac{(N_j-1)\kappa_a\kappa_l}{N_j\kappa_a-\kappa_l}. \tag{66}$$

For a frozen κ_a a discontinuity in $\kappa_{w\,app}$ appears when $\kappa_l = \kappa_a$, however, the discontinuity for $\kappa_{b\,app}$ takes place when $\kappa_l = N_j\kappa_a$.

The estimates of the within- and between-sites scatter obtained from equation (65) and (66) have been compared, for different values of κ_a and κ_l with those based on solutions of equation (62). For instance, for the fixed values $\kappa_a = 10$ and $N_j = 5$, κ_l is according to relation (63) restricted to the interval (10, 46). Near the lower discontinuity point ($\kappa_l = 11$) κ_b and $\kappa_{b\,app}$ agree very well (to within 0·1), but the deviation ($\kappa_{b\,app}$) increases rapidly towards the upper discontinuity point and becomes 6 for $\kappa_l = 30$ and over a 1000 when $\kappa_l = 45$. The two disregarded solutions of κ_b in equation (62) both remain almost constant and close to zero (−1·8 and 0·61). The deviation between κ_w and $\kappa_{w\,app}$ on the other hand is less extensive and increases towards the lower discontinuity point, where it is about 8 for $\kappa_l = 11$.

The above method of estimating κ_w and κ_b will be referred to as the 'Overall dispersion' method to distinguish it from the 'Circular Standard Deviation' method. By this procedure an estimate of κ_w is obtained from the following equation

$$\kappa_w = 6561/\overline{\theta_{63}^2} \quad (\theta_{63} \text{ in degrees}) \tag{67}$$

here $\overline{\theta_{63}^2}$ symbolizes the mean square of the circular standard deviation of sample means in the studied sites. Either equation (39) or (42) may then be used to complete the analysis. In principle the latter corresponds to the 'Successive method' introduced by Sanver (1968).

8. Examples

A statistical program developed by Gidskehaug (1975) makes use of the above theory. On the basis of stable specimen directions it estimates the overall distribution of remanent magnetization in the rock formation concerned with unit weight on specimens, samples and sites respectively. Further it optionally produces computer graphs of the observed distributions at the same three levels, prints tables suitable for Goodness of Fit tests and finally carries out analysis of dispersion.

Table 2 shows the analysis of dispersion summary table, which includes the value of the F-statistic and the related degrees of freedom. Then follow the number of rock samples (N) and sites (B) dealt with. The estimates of κ_a and κ_l, found previously by equation (12), are required in connection with the Overall dispersion and Circular standard deviation methods. Equations (59) and (60) form the basis for the Mean square method of estimating the within κ_w and between-sites scatter κ_b, equations (67) and (39) for the Circular standard deviation method and equations (39) and (42) for the Overall dispersion method. In each case the estimated dispersion of means of sample means k_{ma} and means of site means k_{ml} originate from equations (45) and (47), while the circular standard deviation (S_{63}) and the 95 per cent confidence circles (A_{95}) are derived from equations (5) and (6) respectively (note that the precise formula (4) has to be used for low precision parameters). For comparison the program uses the derived values of k_w and k_b to recalculate the statistics k_a and k_l by means of equations (39) and (42). The agreement of k_a estimated on the basis of the Mean square method with the overall dispersion of sample means determined more directly by means of equation (12) gives some confidence in the new set of formulae. In case of the Overall dispersion method, k_a and k_l naturally become equal to the corresponding initial values, while generally only one of the two quantities agrees exactly in the Circular standard deviation method depending on whether equation (39) or (42) is used to estimate κ_b.

Analyses of dispersion carried out on palaeomagnetic results from the Kaoko Lavas in South-West Africa (Gidskehaug *et al.* 1975) are presented in Table 2. The input includes stable remanent magnetization directions from a total of 118 rock samples collected from 40 sites. The size 3·77 of the test variable F with the listed degrees of freedom suggests that a between-site effect does exist.

In this example the Overall Dispersion method yields the biggest estimate of κ_w of the three methods and the smallest value κ_b, but the difference in the corresponding circular standard deviation is smaller than one degree. Finally the confidence circles A_{95} of the mean of sample means agree exactly (with an accuracy of at least two decimals) while the estimated uncertainties of the mean of sites means deviate from each other by about 0·1 degrees.

Table 2

Analysis of dispersion of palaeomagnetic results from the Kaoko lavas in South-West Africa.

Source	Degree of Freedom	Sum of square	Mean square
Between sites	78	2·0162	0·02585
Within sites	156	1·0710	0·00687

$F = 3{\cdot}77$ with 78 and 156 df

$N = 118 \quad B = 40 \quad k_a = 37{\cdot}90 \quad k_l = 53{\cdot}26$

	Mean square method		Circular sd. method		Overall dispersion method	
Within-site dispersion:	$k_w = 72{\cdot}83$	$S_{63} = 9{\cdot}49$	$k_w = 79{\cdot}25$	$S_{63} = 9{\cdot}10$	$k_w = 87{\cdot}69$	$S_{63} = 8{\cdot}65$
Between-site dispersion:	$k_b = 80{\cdot}12$	$S_{63} = 9{\cdot}05$	$k_b = 73{\cdot}56$	$S_{63} = 9{\cdot}44$	$k_b = 67{\cdot}52$	$S_{63} = 9{\cdot}86$
Overall dispersion of sample means:	$k_a = 37{\cdot}90$		$k_a = 37{\cdot}90$		$k_a = 37{\cdot}90$	
Overall dispersion of site means:	$k_l = 57{\cdot}96$		$k_l = 55{\cdot}61$		$k_l = 53{\cdot}26$	
Dispersion of means of sample means:	$k_{ma} = 4355{\cdot}07$	$A_{95} = 2{\cdot}12$	$k_{ma} = 4355{\cdot}06$	$A_{95} = 2{\cdot}12$	$k_{ma} = 4355{\cdot}06$	$A_{95} = 2{\cdot}12$
Dispersion of means of site means:	$k_{ml} = 2279{\cdot}34$	$A_{95} = 2{\cdot}93$	$k_{ml} = 2185{\cdot}59$	$A_{95} = 2{\cdot}99$	$k_{ml} = 2091{\cdot}21$	$A_{95} = 3{\cdot}06$

9. Discussion

Creer (1962a) and Irving (1964) discuss various sources that might contribute to the scatter of the remanent magnetization in rocks. In general, experimental errors due to orientation and measurement in addition to inhomogeneity and anisotropy of magnetization will contribute to the within-site scatter. Further, the between-sites dispersion may in part result from tectonic tilting, but when allowing for this effect, the remaining scatter is often attributed to the palaeosecular variation of the Earth's magnetic field. The secular variation of the present geomagnetic field appears to be latitude dependent (Creer 1962a; Cox 1970). Assuming the same to apply in the geological past, then κ_b or the corresponding circular standard deviation θ_{63} serves as a measure of the amount of secular variation at a particular palaeolatitude. The procedure requires that the true mean of stable remanent magnetization within a site consists of a spot reading or has been acquired at a relatively short interval of time, while the time spread between sites extends sufficiently long to record the secular effect. A pile of lava flows could originally meet these requirements. However, remagnetization or the acquisition of stable secondary components could easily overprint a primary between-sites effect. In this connection Storetvedt (1968) claims that a VRM built up over a long time at moderate temperatures could replace the original remanence such that a site records the secular variation and its mean represent an axial dipole field. Disregarding polar wandering and tectonic tilting one would under these circumstances expect no between-sites scatter or $\kappa_b \gg \kappa_w$, thus allowing the hypothesis to be tested by analysis of dispersion. Precaution must, however, be taken to ensure that a sufficiently long time is involved, since the condition $\kappa_b \gg \kappa_w$ could also arise primarily from a rapidly erupted sequence of flows.

Equation (42) and the corresponding approximate formula (49) show that the sampling distribution of site mean directions mask the true between-sites scatter κ_b, since the precision parameter κ_l depends also on the within-site scatter κ_w, resulting in $\kappa_l < \kappa_b$. For this reason an estimate of κ_l should not be used as a measure of the between-sites scatter. Despite this the observed overall scatter of site means k_l has frequently been used to estimate the palaeosecular variation (κ_b). This was the case with most of the palaeomagnetic results used by Brock (1971) to test the various models (Creer, Irving & Nairn 1959; Creer 1962a; Irving & Ward 1964; Cox 1970) for secular variation of the Earth's magnetic field. Generally, k_l will result in an overestimation of κ_b. However, when $N_j \kappa_w \gg \kappa_b$ (see equation(49)) κ_l will approach κ_b and the assumptions made by Creer (1962a, 1962b) apply.

This paper describes three different methods to isolate or to estimate the between-sites scatter κ_b. The formulae used to obtain the results in Table 2 (see Section 8), will be referred to as 'exact' to distinguish them from the approximate equations (61), (48) and (49). The statistical program has the ability to carry out analysis of dispersion also by the latter. As expected, due to the high precision of the remanent magnetization in the Kaoko lavas, the different procedures in this case agree very well. A replacement of equation (60) by (61) in the Mean square method reduces the estimated between-sites scatter to $k_b = 77{\cdot}4$ or $S_{63} = 9{\cdot}2$ as quoted by Gidskehaug *et al.* (1975). Frequently, the estimates k_a and k_l of the overall dispersion of sample means and site means, respectively, are quoted in palaeomagnetic papers and thus supply the main information to estimate the within- and between-sites scatter by means of equations (65) and (66). With respect to the Overall dispersion method the deviation between k_b and $k_{b\,app}$ does not simply depend on the absolute values of k_a and k_l but also of their relative configuration as discussed at the end of Section 7.

The estimated Fisherian statistics apply only if a studied population conforms to a Fisherian distribution. This may be tested by a Goodnes of fit test (Watson & Irving 1957). The assumption made to deduce equation (39) are given by the equations (34), (35) and (36) in addition to the requirement that the random vectors ϖ_{ij} and β_j must

be independent of each other. The same assumptions underlie the statistical induction about the between- and within-site scatter in the analysis of dispersion. Equation (34) implies that a sample mean $\mathbf{p}_{ij}$, randomly chosen from a site, must be Fisherian distributed with constant precision parameter κ_w. To test the B sites for homogeneity in dispersion, Hartley's maximum F-ratio test may be applied (Watson 1956; Pearson & Hartley 1970). Preferably, also the number of rock-samples N_j used in a site should be equal for all sites. Although not proved, it is believed that a good agreement between the various methods to estimate κ_w and κ_b indicates that the underlying assumptions have not been severely violated. In case of the Kaoko lavas the overall distribution of sample means and site means conform to a Fisherian distribution (Gidskehaug *et al.* 1975). The F-test (see Section 6) indicates that a between-sites effect does exist and the estimated between-site scatter k_b was attributed to secular variation since tectonic tilting could be ignored.

Fisherian and Gaussian statistics have many points of resemblance. While a Fisherian distribution is uniquely defined by means of the two parameters $\boldsymbol{\mu}$ and κ, a normal distribution is likewise defined by a true mean value μ and a variance σ^2, where for tightly-grouped distributions the latter corresponds to the reciprocal of the precision parameter (Fisher 1953).

$$\sigma^2 = 1/\kappa. \tag{68}$$

By means of the above expression, equations (48), (49), (59) and (51) transform directly to similar equations in Gaussian statistics. This is attributed to the relation between the probability density functions of the two types of distributions. Suppose the independent variable z of the probability density function $g(z)$ of a normal distribution symbolizes the deviation of a random observation X from the true means μ. Then by replacing θ by z in equation (1) we get for large value of κ.

$$f_1(z) = f_1(z, \phi) = (2\pi\sigma^2)^{-\frac{1}{2}} g(z) = (\kappa/2\pi)^{\frac{1}{2}} g(z). \tag{69}$$

Equation (69) shows that the function value $f_1(z) = g(z)$ when $\kappa = 1/\sigma^2 = 2\pi$ and, further, $f_1(z) > g(z)$ if $\kappa > 2\pi$. Notice, however, that g is defined for all real numbers while the surface of a unit sphere constitutes the domain of definition of f_1.

For the case when the true mean direction $\boldsymbol{\mu}$ is known in a Fisherian distribution, it was shown in Section 3 that the mean square $\overline{\omega^2}$ of the vector error $\boldsymbol{\omega}_i = \mathbf{p}_i - \boldsymbol{\mu}$ $(i = 1, 2, \ldots N)$ constitutes an unbiased estimator of $2/\kappa$ (equations (19) and (22)). This estimator has a general application for Fisherian distributions for $\kappa > 3$. For high values of the precision parameter we may in addition estimate κ by means of the angular deviation θ_i (in radians) between $\boldsymbol{\mu}$ and $\mathbf{p}_i$.

$$2/k = \sum^{N} \theta_i^2/N = \overline{\theta^2}. \tag{70}$$

It is interesting to notice that $\overline{\omega^2}$ (or $\overline{\theta^2}$ for large κ) also estimates the square of the radius $\theta_{63\cdot21}$ of the circle within which an observation falls with probability 63·21 per cent (similar to equation (5)) since

$$\theta^2_{63\cdot21} = 2/\kappa = 2\sigma^2 \qquad (\theta_{63\cdot21} \text{ in radians}) \tag{71}$$

Equation (71) shows that $\theta^2_{63\cdot21}$, which is referred to as the angular variance, has a value twice as large as the variance σ^2 in the corresponding normal distribution (equation (68)).

For the case when $\boldsymbol{\mu}$ is unknown, we see from equations (25) and (71) that $\sum \delta_i^2/N-1$ is an unbiased estimator of the angular variance $\theta^2_{63\cdot21}$. This is similar to the unbiased estimator $\hat{s}^2$ of the variance σ^2 in a normal distribution

$$\hat{s}^2 = \sum^{N} d_i^2/N-1 \tag{72}$$

where d_i symbolizes the deviation of a random observation X_i from the sample mean $M = \sum^{N} X_i/N$. Only when applied to a tightly-grouped Fisherian distribution (large κ) formula (1) given by Cox (1970) constitutes an unbiased estimator of the angular variance $(2/\kappa)$, because then the angular deviation becomes equal to the corresponding vector deviation δ_i.

10. Conclusion

The domain of definition of the basic probability density function (1) of a Fisherian distribution forms a unit sphere. An element in this domain may be referred to in a series of ways among which the vector error $\boldsymbol{\omega}_i = (\mathbf{p}_i - \boldsymbol{\mu})$ or the vector deviation from the true direction $\boldsymbol{\mu}$ is introduced in this paper. If a sample of N random observations $\mathbf{p}_i$ has been taken from a Fisherian distribution for which $\boldsymbol{\mu}$ is known, then twice the reciprocal mean square of the vector errors makes an unbiased estimator of the precision parameter κ. This follows from the expectation $E(\omega^2)$ of the square of $\boldsymbol{\omega}_i$ deduced from equation (3) or ω_i marginal distribution of $c = \cos\theta$. Further, the expectation $E(R)_{4,0}$ given by equation (33), where R symbolizes the scalar value of the vector sum of the N unit vectors $\mathbf{p}_i$, conforms to the linear function (29) of N for the whole range of $\kappa > 3$ and for N values up to 3κ. On these assumptions an unbiased estimator of κ is derived also for the general case when $\boldsymbol{\mu}$ is unknown. The latter turned out to be identical to the estimate of precision given by Fisher, while the formula (in terms of vector deviation) for estimating the corresponding angular variance is similar to the unbiased estimate of variance in Gaussian statistics.

Based on the above information and a one way random effect vector model the overall dispersion of sample means and site means are derived as functions of the within- and between-sites scatter. These formulae, which take the curvature of the sphere into account, transform for large precision parameters to the corresponding formulae in Gaussian statistics by substituting $\kappa = 1/\sigma^2$.

The analysis of dispersion on a sphere is generalized to include populations with precision parameters (κ_a) down to about three and sample sizes (N) at least up to three times the precision parameter. A new formula for the expectation of the mean square between sites improves the Mean square method. In addition, the Circular standard deviation method and Overall dispersion method of estimating the within- and between-sites scatter are described.

Finally, formulae for the dispersion of the sampling distribution of means of sample means and means of site means are used to estimate the uncertainty of overall mean values. These provide a more general application of the theory of errors to distributions of directions in space.

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An Iterative Approximation to the Mixed-Signal Proce

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Summary

An iterative array processor for two simultaneously arriving signals is developed which gives estimates equal to the maximum likelihood estimates. Using this processor, the array is first beamed on one of the two events to produce a signal estimate which is then time-shifted and subtracted from each of the original traces. The difference traces are then beamed to produce a signal estimate for the second event. The estimate for the second event is now shifted and subtracted from the *original* traces, and the resulting difference traces are rebeamed on the first event. The process is repeated until differences in successive signal estimates for the desired event fall below a pre-determined threshold. In addition to its use in processing two simultaneously arriving signals, the processor can be of use for detection of secondary phases in the coda of an event.

Introduction

The most direct approach for separating two signals which arrive simultane at a seismic array is to beam the array on each of the event epicentres. Unfortun the simple beam does not always yield satisfactory signal estimates due to conta tion of one signal's estimate by the other. Shumway & Dean (1968) and C (1974), however, demonstrated that an asymptotic maximum-likelihood proc yields better signal estimates than does simple beamforming. Cohen, using si recorded at the Tonto Forest Seismic Observatory (TFO), found that the attenuation obtained using a seven-element subarray and the mixed-signal proc was comparable to that obtained using a 19-element subarray and the beam.

While the superiority of the mixed-signal processor has been established by and other studies, for computational reasons it is not always practical to us maximum-likelihood approach. However, the iterative beamforming appr described here is similar in concept to simple beamforming, and it may be po to implement in operational systems with minimal investments in addit programming.

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