(Text continued on inside back cover)

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THE STREAM FUNCTION AND GAUSS' PRINCIPLE OF LEAST CONSTRAINT: TWO USEFUL CONCEPTS FOR STRUCTURAL GEOLOGY

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ABSTRACT

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To the extent that rock deformation can be approximated by a two-dimensional Newtonian model, a powerful stream-function simulation method is applicable. The significance of stream functions is that velocity, strain, stress and energy derived from the same stream function satisfy automatically three basic conditions of dynamics: (1) the condition of continuity, (2) the Navier-Stokes equations, and (3) conservation of energy. Hence we state with Jaeger: "If a stream function can be found which satisfies the boundary conditions of a dynamic model the complete solution follows." All pertinent bits of dynamic information are implied in the stream function from which they can be directly derived, guaranteed—so to speak—not to violate the basic conditions of dynamics. Stream functions useful in structural geology are solutions of:

$$\nabla^4\psi = \frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4} = 0$$

A double-polynomial solution of max. degree 14 is developed, in which the coefficients are related controlled by the $\nabla^4\psi = 0$ constraint, and their absolute values are determined by the boundary conditions of specific models and by the condition of maximum rate of energy dissipation or maximum rate of decline of potential energy. The polynomial stream function is applied to a collapsing viscous "nappe" consisting of a thin basal layer with low viscosity on which a thicker layer with high viscosity slides due to gravitational spreading. The velocity of forward movement depends upon absolute and relative values of the following parameters: viscosity, thickness, the aspect ratio and density. The velocity of a variety of nappes with different thicknesses, aspect ratios, viscosities and densities is determined.

INTRODUCTION

It is known that the mechanical behaviour of crystalline rocks is infinitely more complex than that of Newtonian fluids; nevertheless, much insight into the evolution of deformation structures of rocks exposed to dynamothermal metamorphism

may be gained from theoretical models in which rocks are treated as just Newtonian fluids, albeit with an extremely high effective viscosity. I believe analyses based on Newtonian models have considerably deepened our understanding of the dynamics of folds and boudinage (Biot, 1959, 1961, 1963; Ramberg, 1968, 1981; Johnson, 1970; Fletcher, 1977; Smith, 1975), of salt- and gneiss domes (Biot and Ode, 1965; Ramberg, op. cit.), of mantle diapirism and of thrust sheets (Elliott, 1976; Price, 1973; Ramberg, 1981) and not least of isostatic adjustment (Haskel, 1935; Crittenden, 1963; Cathles, 1975; Ramberg, 1968; Artyushkov, 1971).

In this paper we shall continue to treat rocks as Newtonian bodies with high effective viscosity when exposed to deviatory stress and the force of gravity in and on the Earth's crust.

The aim of the paper is to demonstrate how a well-known concept in fluid dynamics—the Stream Function—can be combined with another well-known physical principle—Gauss' Principle of Least Constraint—to give information on the evolution of rock structures, including the velocity at which deformation structures develop.

The special structure treated in the paper is a set of composite nappes with dissimilar geometric dimensions and viscosities.

To apply the Stream Function Method it is first necessary to develop a general solution of the biharmonic equation:

$$\partial^4\psi/\partial x^4 + 2\partial^4\psi/\partial x^2\partial y^2 + \partial^4\psi/\partial y^4 = 0$$

This is done in the section "A useful double-polynomial stream function", p. 210. It is further necessary to develop special solutions valid for the particular models considered; in our case composite nappes. The development of special solutions is treated in the section "Coefficient determination by the method of extremizing the rate of energy change", p. 216. Here it is shown how Gauss' classic Principle of Least Constraint (recast in the form of the Principle of Extreme Rate of Energy Change) is used to determine arbitrary coefficients in the polynomial stream function. Coefficients in the stream function valid for the double-layer nappe are determined in the section "Simulation of a spreading composite nappe", p. 218.

The most significant results from the numerical simulation of composite nappes are the velocities and their relation to layer-thickness and length (aspect ratio), to viscosity and viscosity ratio of the two layers as well as to their density. These relationships are presented in the illustrations in Figs. 4–15, which actually contain all pertinent information obtained by the simulation procedure. Readers not interested in the theoretical part of the paper may therefore find useful data concerning nappe motion by studying the illustrations and their text.

THE STREAM FUNCTION METHOD

The assumption that rocks behave during regional metamorphism as extremely viscous Newtonian fluids, combined with the knowledge that inertia is insignificant for slow tectonic processes, enables us to apply a powerful stream-func to the evolution of a number of deformation structures encountered in

Any two-dimensional motion can be described by a stream functio 1781; Rankin, 1864; Lamb, 1932), ψ, which for non-inertial Newtoni called creeping motion—is defined as a solution of the biharmoni equation:

$$\nabla^4\psi = \partial^4\psi/\partial x^4 + 2\partial^4\psi/\partial x^2\partial y^2 + \partial^4\psi/\partial y^4 = 0$$

The stream function is related to the velocity components by the expr

$$u = -\partial\psi/\partial y$$

and:

$$v = \partial\psi/\partial x$$

Here u is the velocity component in the horizontal direction x, and component in the vertical direction y.

The significance of the stream-function method is based on the c solutions of the biharmonic equation are functions (by definition strea whose derived velocity, strains and stresses automatically satisfy thre dynamic relationships, viz. (1) the condition of continuity, (2) the equation of motion, and (3) conservation of energy.

For two-dimensional flow of incompressible fluids the relation:

$$\partial u/\partial x + \partial v/\partial y = 0$$

expresses the condition of continuity. From the definition $u = -\partial\psi$ $\partial\psi/\partial x$ follow $\partial u/\partial x = -\partial^2\psi/\partial y\partial x$ and $\partial v/\partial y = \partial^2\psi/\partial x\partial y$, and mediately that continuity is automatically satisfied in models define function.

For non-inertial viscous flow in two dimensions the Navier-Stoke motion take the form:

$$\partial P/\partial x - \eta\left(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2\right) = 0$$

$$\partial P/\partial y - \eta\left(\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2\right) + \rho g = 0$$

Since v is the velocity component in the vertical direction y, the grav included in eqn. (6).

Differentiation of all terms in eqn. (5) with respect to y and all ter with respect to x yields:

$$\partial^2 P/\partial x\partial y - \eta\left(\partial^3 u/\partial x^2\partial y + \partial^3 u/\partial y^3\right) = 0$$

$$\partial^2 P/\partial y\partial x - \eta\left(\partial^3 v/\partial x^3 + \partial^3 v/\partial y^2\partial x\right) = 0$$

Subtracting eqn. (8) from (7) and applying expressions $\partial\psi/\partial y = -u$ one obtains:

$$\partial^4\psi/\partial x^4 + 2\partial^4\psi/\partial x^2\partial y^2 + \partial^4\psi/\partial y^4 = 0$$

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FUNCTION METHOD

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for slow tectonic processes, enables us to apply a powerful stream-function method to the evolution of a number of deformation structures encountered in the field.

Any two-dimensional motion can be described by a stream function (Lagrange, 1781; Rankin, 1864; Lamb, 1932), ψ, which for non-inertial Newtonial flow—so-called creeping motion—is defined as a solution of the biharmonic differential equation:

$$\nabla^4\psi = \partial^4\psi/\partial x^4 + 2\partial^4\psi/\partial x^2\partial y^2 + \partial^4\psi/\partial y^4 = 0 \tag{1}$$

The stream function is related to the velocity components by the expressions:

$$u = -\partial\psi/\partial y \tag{2}$$

and:

$$v = \partial\psi/\partial x \tag{3}$$

Here u is the velocity component in the horizontal direction x, and v the velocity component in the vertical direction y.

The significance of the stream-function method is based on the condition that solutions of the biharmonic equation are functions (by definition stream functions), whose derived velocity, strains and stresses automatically satisfy three crucial fluid dynamic relationships, viz. (1) the condition of continuity, (2) the Navier-Stokes equation of motion, and (3) conservation of energy.

For two-dimensional flow of incompressible fluids the relation:

$$\partial u/\partial x + \partial v/\partial y = 0 \tag{4}$$

expresses the condition of continuity. From the definition $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$ follow $\partial u/\partial x = -\partial^2\psi/\partial y\partial x$ and $\partial v/\partial y = \partial^2\psi/\partial x\partial y$, and one sees immediately that continuity is automatically satisfied in models defined by a stream function.

For non-inertial viscous flow in two dimensions the Navier-Stokes equations of motion take the form:

$$\partial P/\partial x - \eta\left(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2\right) = 0 \tag{5}$$

$$\partial P/\partial y - \eta\left(\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2\right) + \rho g = 0 \tag{6}$$

Since v is the velocity component in the vertical direction y, the gravity term ρg is included in eqn. (6).

Differentiation of all terms in eqn. (5) with respect to y and all terms in eqn. (6) with respect to x yields:

$$\partial^2 P/\partial x\partial y - \eta\left(\partial^3 u/\partial x^2\partial y + \partial^3 u/\partial y^3\right) = 0 \tag{7}$$

$$\partial^2 P/\partial y\partial x - \eta\left(\partial^3 v/\partial x^3 + \partial^3 v/\partial y^2\partial x\right) = 0 \tag{8}$$

Subtracting eqn. (8) from (7) and applying expressions $\partial\psi/\partial y = -u$ and $\partial\psi/\partial x = v$ one obtains:

$$\partial^4\psi/\partial x^4 + 2\partial^4\psi/\partial x^2\partial y^2 + \partial^4\psi/\partial y^4 = 0$$

which is the biharmonic base for stream functions. Hence the equations of motion are automatically satisfied in models defined by stream functions.

It is particularly important to realize that energy is also automatically conserved when expressed by formulas derived from one and the same stream function. This gives crucial support to the validity of the energy-extremizing method of determining coefficients in a polynomial stream function (p. 216).

To demonstrate that energy is automatically conserved when derived from a stream function requires quite a lengthy mathematical procedure, so let it suffice here to merely indicate how the demonstration may be performed.

For a deforming viscous model described by a stream function there are several categories of mechanical energy to consider, and all are derivable from the stream function used. Expressed in terms of rate of change of energy per unit volume (for strain energy and potential energy) or per unit surface area (for energy due to stress at the boundary), the different categories are:

$\dot{e}_\epsilon = 4\eta\dot{\epsilon}_\phi^2$, the rate of change of normal-strain energy per unit volume; (9)

$\dot{e}_\gamma = \eta\dot{\gamma}_\phi^2$, the rate of change of shear-strain energy per unit volume; (10)

$\dot{e}_{\text{pot}} = \rho g v$, the rate of change of gravitational energy per unit volume; (11)

$\dot{e}_\tau = \tau_t v_t$, the rate of energy input or output per unit area due to shear stress at the boundary; and finally (12)

$\dot{e}_\sigma = \sigma_n u_n$, the rate of energy input or output per unit area due to normal stress at the boundary. (13)

In these expressions subscript ϕ indicates direction in space, v is the velocity component in the vertical direction, τ_t is shear stress parallel to the boundary, v_t is the velocity at the boundary parallel to the shear stress, σ_n is normal stress at the boundary and u_n is the velocity component normal to the boundary.

To obtain the energy changes for the whole model the quantities $\dot{e}_\epsilon$, $\dot{e}_\gamma$ and $\dot{e}_{\text{pot}}$ must be integrated over the volume occupied by the model and $\dot{e}_\tau$ and $\dot{e}_\sigma$ must be integrated over the boundary. Using capital letters for the integrated energies we combine some of the energies specified above:

$$\dot{E}_{\epsilon\gamma} = \dot{E}_\epsilon + \dot{E}_\gamma \tag{14}$$

$$\dot{E}_{\sigma\tau} = \dot{E}_\sigma + \dot{E}_\tau \tag{15}$$

The crucial point is that when $\dot{E}_{\epsilon\gamma}$, $\dot{E}_{\sigma\tau}$ and $\dot{E}_{\text{pot}}$ are derived from one and the same stream function, we find that the condition of conservation is automatically satisfied, viz:

$$\dot{E}_{\epsilon\gamma} + \dot{E}_{\sigma\tau} + \dot{E}_{\text{pot}} = 0 \tag{16}$$

In other words, energy—as expressed by formulas derived from a stream function—is being conserved during the strain and motion which occur in the model.

In order to derive the energies the following formulas are used:

$$\dot{\epsilon}_x^2 = \left(\frac{\partial u}{\partial x}\right)^2 = \left(-\frac{\partial^2\psi}{\partial x\partial y}\right)^2, \quad \dot{\epsilon}_y^2 = \left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2$$

$$\dot{\gamma}_{xy}^2 = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 = \left(-\frac{\partial^2\psi}{\partial y_2} + \frac{\partial^2\psi}{\partial x^2}\right)^2$$

$\tau_{xy} = \eta\dot{\gamma}_{xy}$ and $\sigma_x = 2\eta\dot{\epsilon}_x - P$

together with the derived velocity components.

To form P, the general formula:

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$$

is applicable. Here we introduce:

$$\frac{\partial P}{\partial x} = \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = -\eta\left(\frac{\partial^3\psi}{\partial x^2\partial y} + \frac{\partial^3\psi}{\partial y^3}\right)$$

and:

$$\frac{\partial P}{\partial y} = \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \rho g = \eta\left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\psi}{\partial x\partial y^2}\right) - \rho g$$

Again, the last two equations are directly derived from the stream fu

In order to obtain P (and thus also σ) by integration of eqn. (20 realize that dP is an exact differential. This can be shown when the

$$\partial\left(\frac{\partial P}{\partial x}\right)/\partial y = \partial\left(\frac{\partial P}{\partial y}\right)/\partial x$$

is applied in combination with the biharmonic $\nabla^4\psi = 0$.

When the above formulas are applied to a stream function descri ing model of Newtonian material the condition of conservation energy expressed in eqn. (16) follows.

In accordance with the explanation above, we conclude with J 140): "If a stream function can be found which satisfies the bounda a dynamic model the complete solution follows." The stream func velocity field, which in turn implies the strain rate and accordingl coefficient and density—also implies the stress distribution and th dissipation due to viscous strain, as well as the rate of change of p Also the rate of energy input/output due to stress acting on movi implied in the stream function when viscosity and density are given. *the real problem is to find the appropriate stream function; to e pertinent information is mostly routine use of standard fluid dynamic*

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$$\dot{\gamma}_{xy}^2 = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 = \left(-\frac{\partial^2\psi}{\partial y_2} + \frac{\partial^2\psi}{\partial x^2}\right)^2 \tag{18}$$

$$\tau_{xy} = \eta\dot{\gamma}_{xy} \text{ and } \sigma_x = 2\eta\dot{\epsilon}_x - P \tag{19}$$

together with the derived velocity components.

To form P, the general formula:

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy \tag{20}$$

is applicable. Here we introduce:

$$\frac{\partial P}{\partial x} = \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = -\eta\left(\frac{\partial^3\psi}{\partial x^2 \partial y} + \frac{\partial^3\psi}{\partial y^3}\right) \tag{21}$$

and:

$$\frac{\partial P}{\partial y} = \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \rho g = \eta\left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\psi}{\partial x \partial y^2}\right) - \rho g \tag{22}$$

Again, the last two equations are directly derived from the stream function.

In order to obtain P (and thus also σ) by integration of eqn. (20) it is useful to realize that dP is an exact differential. This can be shown when the criterium:

$$\partial\left(\frac{\partial P}{\partial x}\right)/\partial y = \partial\left(\frac{\partial P}{\partial y}\right)/\partial x$$

is applied in combination with the biharmonic $\nabla^4\psi = 0$.

When the above formulas are applied to a stream function describing a deforming model of Newtonian material the condition of conservation of mechanical energy expressed in eqn. (16) follows.

In accordance with the explanation above, we conclude with Jaeger (1956, p. 140): "If a stream function can be found which satisfies the boundary conditions of a dynamic model the complete solution follows." The stream function implies the velocity field, which in turn implies the strain rate and accordingly—via viscosity coefficient and density—also implies the stress distribution and the rate of energy dissipation due to viscous strain, as well as the rate of change of potential energy. Also the rate of energy input/output due to stress acting on moving boundaries is implied in the stream function when viscosity and density are given. In other words, *the real problem is to find the appropriate stream function; to extract the other pertinent information is mostly routine use of standard fluid dynamic relationships.*

We note that stream functions give the instantaneous velocities, the instantaneous strain rates, the instantaneous stress distribution etc. The evolution in finite

time of a model accordingly depends upon integration problems not related to the question of finding the stream function itself. However, even if integration over a finite time interval may pose formidable and perhaps even insoluble problems, quantitative knowledge of the instantaneous velocity, stress etc. is itself of great value for the understanding of the evolution of rock structures. See also p. 239.

For periodic structures such as, for example, bucklefolds in layered rocks and periodic series of domes and diapirs, harmonic stream functions of type:

$$\psi = \exp(\pm\omega y)[A\cos(\omega x) \pm B\sin(\omega x)] \tag{23}$$

$$\psi = y\exp(\pm\omega y)[A\cos(\omega x) \pm B\sin(\omega x)] \tag{24}$$

($\omega = 2\pi/\lambda$, λ is wavelength)

both being solutions of $\nabla^4\psi = 0$—are applicable and yield useful results such as the dominant wavelength of folds, the spacing between diapirs, the rate of isostatic adjustment etc. as long as the amplitude is small compared with the wavelength. This is demonstrated and thoroughly discussed in some of the works cited above (e.g., Ramberg, 1968, 1981) and will not be repeated here. Instead we shall consider polynomial forms of stream functions, which are applicable to various non-periodic geologic structures.

A USEFUL DOUBLE-POLYNOMIAL STREAM FUNCTION

The application of a polynomial stream function to the slow "creeping" gravitational spreading of a viscous nappe was published in Ramberg (1981, pp. 267–226). (Unfortunately a numerical error affected the calculated strain energy of the model, but a correction has been published, see Ramberg, 1985.)

In the cited book (Ramberg, 1981) the stream function was applied to a simple nappe model with a rather special velocity field. The nappe was coherent to the rigid basement, and it was symmetric about a vertical plane through its center. This means that the velocity components u and v vanish at the base, that u vanishes also at the central cross section and that the stream function developed was not useable for nappes with more realistic boundary conditions.

In the present study we shall develop a polynomial stream function which is versatile enough to simulate a variety of non-periodic structures, including nappes with non-vanishing velocity at the base. Nappes described by stream functions of this nature are not only permitted to slide along the base while spreading gravitationally, but also allow a "push from behind". Incidentally, it is interesting to note that the double-polynomial function to be developed (eqn. 31) is comprehensive enough to contain the periodic solutions (23) and (24). This can be demonstrated by appropriate adjustment of the arbitrary coefficients using Taylor-Maclaurin series for $\sin(\omega x)$, $\cos(\omega x)$, $\exp(\omega y)$.

To develop the general polynomial stream function it is convenient to start by considering the velocity component u and its variation with y at distance x from

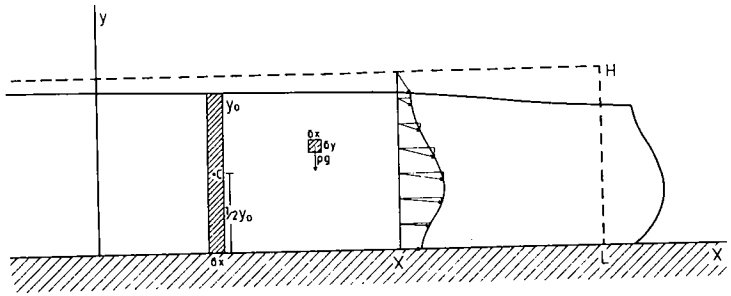

Fig. 1. Right-hand half of viscous body with initially rectangular cross section paral with infinite length in z. Initial cross section dashed, profile after arbitrary time of de solid lines. Displacement of vertical straight marker at x indicated by velocity vecto

the central plane (Fig. 1). As the material of the nappe is s homogeneous and Newtonian, u is evidently a continuous and smoo at constant x. Hence u can be expressed as a polynomial in y:

$$u = -\left[a + 2by + 3cy^2 + 4dy^3 + \ldots + nwy^{(n-1)}\right]$$

(cf. Weierstrass' theorem which states that any continuous funct proximated by a polynomial—Courant, 1953, p. 423).

The reason for the negative sign and the factors 2, $3\ldots n$ coefficients is to obtain a positive and simple form of the corres function—see below. In polynomial (25) the coefficients are unkno x, and each of the coefficients may be expressed in polynomial fori

$$a = a_1 + a_2x + a_3x^2 + \ldots + a_mx^{(m-1)}$$

$$b = b_1 + b_2x + b_3x^2 + \ldots + b_mx^{(m-1)}$$

$$c = c_1 + c_2x + c_3x^2 + \ldots + c_mx^{(m-1)}$$

$$\vdots$$

$$w = w_1 + w_2x + w_3x^2 + \ldots + w_mx^{(m-1)}$$

Using the definition $u = -\partial\psi/\partial y$ we form a function ψ by inte

$$\psi = ay + by^2 + cy^3 + dy^4 + \ldots + wy^n + f(x)$$

Here $f(x)$ is a function of x alone. From the definition $v = \partial\psi/\partial$ form of ψ follow that $f(x)$ determines the velocity component v a reasonable to express $f(x)$ as a polynomial in x:

$$f(x) = \alpha_1 + \alpha_2x + \alpha_3x^2 + \ldots + \alpha_mx^{(m-1)}$$

(Clearly, eqn. (27) as it stands is not a stream function unless th correlated in accordance with the constraint $\nabla^4\psi = 0$. The result c is shown below, p. 214–215).

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ed book (Ramberg, 1981) the stream function was applied to a simple l with a rather special velocity field. The nappe was coherent to the rigid nd it was symmetric about a vertical plane through its center. This he velocity components u and v vanish at the base, that u vanishes also l cross section and that the stream function developed was not useable vith more realistic boundary conditions.

esent study we shall develop a polynomial stream function which is ugh to simulate a variety of non-periodic structures, including nappes ıishing velocity at the base. Nappes described by stream functions of re not only permitted to slide along the base while spreading gravita- also allow a "push from behind". Incidentally, it is interesting to note ible-polynomial function to be developed (eqn. 31) is comprehensive ntain the periodic solutions (23) and (24). This can be demonstrated by adjustment of the arbitrary coefficients using Taylor-Maclaurin series $\cos(\omega x)$, $\exp(\omega y)$.

p the general polynomial stream function it is convenient to start by he velocity component u and its variation with y at distance x from

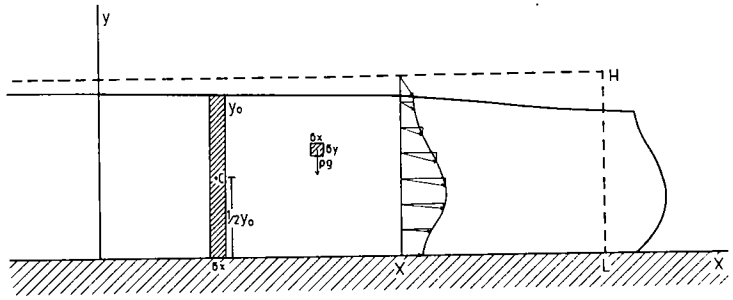

Fig. 1. Right-hand half of viscous body with initially rectangular cross section parallel to plane xy and with infinite length in z. Initial cross section dashed, profile after arbitrary time of deformation shown in solid lines. Displacement of vertical straight marker at x indicated by velocity vectors.

the central plane (Fig. 1). As the material of the nappe is supposed to be homogeneous and Newtonian, u is evidently a continuous and smooth function of y at constant x. Hence u can be expressed as a polynomial in y:

$$u = -\left[a + 2by + 3cy^2 + 4dy^3 + \ldots + nwy^{(n-1)}\right] \quad (25)$$

(cf. Weierstrass' theorem which states that any continuous function can be approximated by a polynomial—Courant, 1953, p. 423).

The reason for the negative sign and the factors 2, 3...n in front of the coefficients is to obtain a positive and simple form of the corresponding stream function—see below. In polynomial (25) the coefficients are unknown functions of x, and each of the coefficients may be expressed in polynomial form:

$$\begin{aligned} a &= a_1 + a_2x + a_3x^2 + \ldots + a_mx^{(m-1)} \\ b &= b_1 + b_2x + b_3x^2 + \ldots + b_mx^{(m-1)} \\ c &= c_1 + c_2x + c_3x^2 + \ldots + c_mx^{(m-1)} \\ &\vdots \\ w &= w_1 + w_2x + w_3x^2 + \ldots + w_mx^{(m-1)} \end{aligned} \quad (26)$$

Using the definition $u = -\partial\psi/\partial y$ we form a function ψ by integration:

$$\psi = ay + by^2 + cy^3 + dy^4 + \ldots + wy^n + f(x) \quad (27)$$

Here $f(x)$ is a function of x alone. From the definition $v = \partial\psi/\partial x$ and the above form of ψ follow that $f(x)$ determines the velocity component v at $y = 0$, and it is reasonable to express $f(x)$ as a polynomial in x:

$$f(x) = \alpha_1 + \alpha_2x + \alpha_3x^2 + \ldots + \alpha_mx^{(m-1)} \quad (28)$$

(Clearly, eqn. (27) as it stands is not a stream function unless the coefficients are correlated in accordance with the constraint $\nabla^4\psi = 0$. The result of this correlation is shown below, p. 214–215).

EQUATION 29

	1	y	y^2	y^3	y^4	y^5	...	$y^{(n-1)}$
1	α_1	a_1	b_1	c_1	d_1	e_1	...	w_1
x	α_2	a_2	b_2	c_2	d_2	e_2	...	w_2
x^2	α_3	a_3	b_3	c_3	d_3	e_3	...	w_3
x^3	α_4	a_4	b_4	c_4	d_4	e_4	...	w_4
x^4	α_5	a_5	b_5	c_5	d_5	e_5	...	w_5
x^5	α_6	a_6	b_6	c_6	d_6	e_6	...	w_6
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮
$x^{(m-1)}$	α_m	a_m	b_m	c_m	d_m	e_m	...	w_m

The function ψ is conveniently presented in the form of a two-dimensional array, eqn. (29).

This arrangement means that each element in the array is the coefficient for the product $x^p y^q$ in which x^p is the multiplier for the row, and y^q the multiplier for the column, in which the element occurs. d_4, for example, is the coefficient for x^3y^4, c_6 the coefficient for x^5y^3, b_5 for x^4y^2 etc., thus $d_4x^3y^4$, $c_6x^5y^3$ and $b_5x^4y^2$ are examples of terms in the polynomial.

When the coefficients of the double polynomial are presented in array form it is desirable to use the more self-explanatory double-index notation expressed in eqn. (30).

In the double polynomial (30) the number of unknown coefficients is $m \cdot n$ when the degree of the polynomial is $(m-1)$ with respect to x and $(n-1)$ with respect to y. We see that the number of unknown coefficients is considerably reduced by the constraint $\nabla^4\psi = 0$. After some cumbersome algebraic and arithmetic operations the correlations recorded in eqn. (31) based on degree 14 in both x and y, are determined.

(To determine the coefficient correlation we derive $\partial^4\psi/\partial x^4$, $\partial^4\psi/\partial y^4$ and $\partial^4\psi/\partial x^2\partial y^2$ from polynomial (30), collect terms with same power of x and same power of y, put each collection of equal-power terms in the form: $2\partial^4\psi/\partial x^2\partial y^2 + \partial^4\psi/\partial y^4$ and equate to zero. From the set of eq obtained the coefficient correlation follows.)

EQUATION 30

	1	y	y^2	y^3	y^4	y^5	...	$y^{(n-1)}$
1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	...	a_{1n}
x	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	...	a_{2n}
x^2	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	...	a_{3n}
x^3	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	...	a_{4n}
x^4	a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	...	a_{5n}
x^5	a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}	...	a_{6n}
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮
$x^{(m-1)}$	a_{m1}	a_{m2}	a_{m3}	a_{m4}	a_{m5}	a_{m6}		a_{mn}

We find that the coefficients in the two first columns and in the tw rows are arbitrary while the remaining coefficients either vanish or are the coefficients in rows 1 and 2, and in columns 1 and 2.

Row 1 constitutes a polynomial in y, and row 2 a polynomial in y x in the first power. Column 1 constitutes a polynomial in x, and polynomial in x multiplied by y in the first power. These four polyno incidentally have the coefficients a_{11}, a_{12}, a_{21}, a_{22} in common—are polynomials whose coefficients are not constrained by the condition are instead free to assume any values determined by the conditions in special model studied.

One notes that the four arbitrary polynomials in the double-power s of the partial differential equation $\nabla^4\psi = 0$ play the same role as constants in series solution of ordinary differential equations (see, e. Tranter, 1961, p. 111 ff.).

In the following we distinguish between the arbitrary coeffici dependent coefficients, the latter being functions of the former.

It is interesting to note that the high-degree cross terms, $a_{ij}x^{(i-1)}$ when $i+j > p+2$ where p is the degree of the arbitrary polynomial: $p = 14$ and we see that all coefficients whose index sum $i+j > 16$, are worth noting that the degree of the arbitrary polynomials in y is th degree of the arbitrary polynomials in x, i.e. $m = n$. This is a c $\nabla^4\psi = 0$.

It is rarely necessary to use the comprehensive polynomial (31) of as 14. Truncated versions will usually do to analyse physical models. are therefore perhaps needed concerning the procedure of truncating altering the polynomial in such a manner that the altered version rem of $\nabla^4\psi = 0$, and hence a valid stream function.

If we look carefully, we note that only coefficients with equ $p = i+j$, are correlated. This means that correlation involves only coe same diagonal running from upper right to lower left. It is also inte one and the same diagonal only coefficients in alternating sites a example, $a_{3.13}$, $a_{5.11}$, a_{79}, a_{97}, $a_{11.5}$ and $a_{13.3}$ are interrelated becau functions of the two arbitrary coefficients $a_{1.15}$ and $a_{15.1}$. Similarly a $a_{10.6}$ and $a_{12.4}$ are interrelated because all are functions of the coefficients $a_{2.14}$ and $a_{14.2}$.

It follows from the first relationship that the polynomial can without violating the constraint $\nabla^4\psi = 0$, by cutting out diagonal af coefficients with a successively smaller index sum, $p = i+j$.

Another way of altering the comprehensive polynomial withou

y	y^2	y^3	y^4	y^5	...	$y^{(n-1)}$
a_1	b_1	c_1	d_1	e_1	...	w_1
a_2	b_2	c_2	d_2	e_2	...	w_2
a_3	b_3	c_3	d_3	e_3	...	w_3
a_4	b_4	c_4	d_4	e_4	...	w_4
a_5	b_5	c_5	d_5	e_5	...	w_5
a_6	b_6	c_6	d_6	e_6	...	w_6
⋮	⋮	⋮	⋮	⋮		⋮
a_m	b_m	c_m	d_m	e_m	...	w_m

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in which the element occurs. d_4, for example, is the coefficient for x^3y^4,
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$^4\psi = 0$. After some cumbersome algebraic and arithmetic operations the
recorded in eqn. (31) based on degree 14 in both x and y, are

nine the coefficient correlation we derive $\partial^4\psi/\partial x^4$, $\partial^4\psi/\partial y^4$ and
from polynomial (30), collect terms with same power of x and same

	y^2	y^3	y^4	y^5	...	$y^{(n-1)}$
12	a_{13}	a_{14}	a_{15}	a_{16}	...	a_{1n}
22	a_{23}	a_{24}	a_{25}	a_{26}	...	a_{2n}
32	a_{33}	a_{34}	a_{35}	a_{36}	...	a_{3n}
42	a_{43}	a_{44}	a_{45}	a_{46}	...	a_{4n}
52	a_{53}	a_{54}	a_{55}	a_{56}	...	a_{5n}
62	a_{63}	a_{64}	a_{65}	a_{66}	...	a_{6n}
	⋮	⋮	⋮	⋮		⋮
m2	a_{m3}	a_{m4}	a_{m5}	a_{m6}		a_{mn}

power of y, put each collection of equal-power terms in the form: $\partial^4\psi/\partial x^4 + 2\partial^4\psi/\partial x^2\partial y^2 + \partial^4\psi/\partial y^4$ and equate to zero. From the set of equations thus obtained the coefficient correlation follows.)

We find that the coefficients in the two first columns and in the two uppermost rows are arbitrary while the remaining coefficients either vanish or are functions of the coefficients in rows 1 and 2, and in columns 1 and 2.

Row 1 constitutes a polynomial in y, and row 2 a polynomial in y multiplied by x in the first power. Column 1 constitutes a polynomial in x, and column 2 a polynomial in x multiplied by y in the first power. These four polynomials—which incidentally have the coefficients a_{11}, a_{12}, a_{21}, a_{22} in common—are the arbitrary polynomials whose coefficients are not constrained by the condition $\nabla^4\psi = 0$, but are instead free to assume any values determined by the conditions imposed by the special model studied.

One notes that the four arbitrary polynomials in the double-power series solution of the partial differential equation $\nabla^4\psi = 0$ play the same role as the arbitrary constants in series solution of ordinary differential equations (see, e.g., Lamb and Tranter, 1961, p. 111 ff.).

In the following we distinguish between the arbitrary coefficients and the dependent coefficients, the latter being functions of the former.

It is interesting to note that the high-degree cross terms, $a_{ij}x^{(i-1)}y^{(j-1)}$, vanish when $i+j > p+2$ where p is the degree of the arbitrary polynomials. In eqn. (31) $p = 14$ and we see that all coefficients whose index sum $i+j > 16$, are zero. It is also worth noting that the degree of the arbitrary polynomials in y is the same as the degree of the arbitrary polynomials in x, i.e. $m = n$. This is a consequence of $\nabla^4\psi = 0$.

It is rarely necessary to use the comprehensive polynomial (31) of as high degree as 14. Truncated versions will usually do to analyse physical models. A few remarks are therefore perhaps needed concerning the procedure of truncating and otherwise altering the polynomial in such a manner that the altered version remains a solution of $\nabla^4\psi = 0$, and hence a valid stream function.

If we look carefully, we note that only coefficients with equal index sum, $p = i+j$, are correlated. This means that correlation involves only coefficients in the same diagonal running from upper right to lower left. It is also interesting that in one and the same diagonal only coefficients in alternating sites are related. For example, $a_{3.13}$, $a_{5.11}$, a_{79}, a_{97}, $a_{11.5}$ and $a_{13.3}$ are interrelated because they are all functions of the two arbitrary coefficients $a_{1.15}$ and $a_{15.1}$. Similarly $a_{4.12}$, $a_{6.10}$, a_{88}, $a_{10.6}$ and $a_{12.4}$ are interrelated because all are functions of the two arbitrary coefficients $a_{2.14}$ and $a_{14.2}$.

It follows from the first relationship that the polynomial can be shortened without violating the constraint $\nabla^4\psi = 0$, by cutting out diagonal after diagonal of coefficients with a successively smaller index sum, $p = i+j$.

Another way of altering the comprehensive polynomial without nullifying the

EQUATION 31

	1	y	y^2	y^3	y^4	y^5	y^6	y^7
1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
x	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}	a_{28}
x^2	a_{31}	a_{32}	$a_{33} = -3(a_{15} + a_{51})$	$a_{34} = -5a_{16} - a_{52}$	$a_{35} = -10a_{17} + 5a_{71}$	$a_{36} = -14a_{18} + a_{72}$	$a_{37} = -21a_{19} - 7a_{91}$	$a_{38} = -27a_{1.10} - a_{92}$
x^3	a_{41}	a_{42}	$a_{43} = -a_{25} - 5a_{61}$	$a_{44} = -\frac{5}{3}(a_{26} + a_{62})$	$a_{45} = -\frac{10}{3}a_{27} + \frac{35}{3}a_{81}$	$a_{46} = -\frac{14}{3}a_{28} + \frac{7}{3}a_{82}$	$a_{47} = -7(a_{29} + 3a_{10.1})$	$a_{48} = -3(3a_{2.10} + a_{10.2})$
x^4	a_{51}	a_{52}	$a_{53} = 5a_{17} - 10a_{71}$	$a_{54} = \frac{35}{3}a_{18} - \frac{10}{3}a_{72}$	$a_{55} = 35(a_{19} + a_{91})$	$a_{56} = 63a_{1.10} + 7a_{92}$	$a_{57} = 126a_{1.11} - 84a_{11.1}$	$a_{58} = 198a_{1.12} - 12a_{11.2}$
x^5	a_{61}	a_{62}	$a_{63} = a_{27} - 14a_{81}$	$a_{64} = \frac{7}{3}a_{28} - \frac{14}{3}a_{82}$	$a_{65} = 7a_{29} + 63a_{10.1}$	$a_{66} = \frac{63}{5}(a_{2.10} + a_{10.2})$	$a_{67} = \frac{126}{5}a_{2.11} - \frac{924}{5}a_{12.1}$	$a_{68} = \frac{198}{5}a_{2.12} - \frac{132}{5}a_{12.2}$
x^6	a_{71}	a_{72}	$a_{73} = -7(a_{19} + 3a_{9.1})$	$a_{74} = -7(3a_{1.10} + a_{9.2})$	$a_{75} = -84a_{1.11} + 126a_{11.1}$	$a_{76} = -\frac{924}{5}a_{1.12} + \frac{126}{5}a_{11.2}$	$a_{77} = -462(a_{1.13} + a_{13.1})$	$a_{78} = -66(13a_{1.14} + a_{13.2})$
x^7	a_{81}	a_{82}	$a_{83} = -a_{29} - 27a_{10.1}$	$a_{84} = -3(a_{2.10} + 3a_{10.2})$	$a_{85} = -12a_{2.11} + 198a_{12.1}$	$a_{86} = -\frac{132}{5}a_{2.12} + \frac{198}{5}a_{12.2}$	$a_{87} = -66(a_{2.13} + 13a_{14.1})$	$a_{88} = -858/7 (a_{2.14} + a_{14.2})$
x^8	a_{91}	a_{92}	$a_{93} = 9a_{1.11} - 36a_{11.1}$	$a_{94} = 33a_{1.12} - 12a_{11.2}$	$a_{95} = 165(a_{1.13} + 2a_{13.1})$	$a_{96} = 66(6.5a_{1.14} + a_{13.2})$	$a_{97} = 1287a_{1.15} - 1716a_{15.1}$	
x^9	$a_{10.1}$	$a_{10.2}$	$a_{10.3} = a_{2.11} - 44a_{12.1}$	$a_{10.4} = \frac{11}{3}(a_{2.12} - 4a_{12.2})$	$a_{10.5} = \frac{55}{3}(a_{2.13} + 26a_{14.1})$	$a_{10.6} = \frac{143}{3}(a_{2.14} + 2a_{14.2})$		
x^{10}	$a_{11.1}$	$a_{11.2}$	$a_{11.3} = -11(a_{1.13} + 5a_{13.1})$	$a_{11.4} = -\frac{55}{3}(2.6a_{1.14} + a_{13.2})$	$a_{11.5} = 286(-a_{1.15} + 2.5a_{15.1})$			
x^{11}	$a_{12.1}$	$a_{12.2}$	$a_{12.3} = -(a_{2.13} + 65a_{14.1})$	$a_{12.4} = -\frac{13}{3}(a_{2.14} + 5a_{14.2})$				
x^{12}	$a_{13.1}$	$a_{13.2}$	$a_{13.3} = 13(a_{1.15} - 6a_{15.1})$					
x^{13}	$a_{14.1}$	$a_{14.2}$						
x^{14}	$a_{15.1}$							

y^8	y^9	y^{10}	y^{11}	y^{12}	y^{13}
a_{19}	$a_{1.10}$	$a_{1.11}$	$a_{1.12}$	$a_{1.13}$	$a_{1.14}$
a_{29}	$a_{2.10}$	$a_{2.11}$	$a_{2.12}$	$a_{2.13}$	$a_{2.14}$
$a_{39} = -36a_{1.11} + 9a_{11.1}$	$a_{3.10} = -44a_{1.12} + a_{11.2}$	$a_{3.11} = -55a_{1.13} - 11a_{13.1}$	$a_{3.12} = -(65a_{1.14} + a_{13.2})$	$a_{3.13} = 13(-6a_{1.15} + a_{15.1})$	
$a_{49} = -12a_{2.11} + 33a_{12.1}$	$a_{4.10} = \frac{11}{3}(-4a_{2.12} + a_{12.2})$	$a_{4.11} = -\frac{55}{3}(a_{2.13} + 2.6a_{14.1})$	$a_{4.12} = -\frac{13}{3}(5a_{2.14} + 10a_{14.2})$		
$a_{59} = 165(2a_{1.13} + a_{13.1})$	$a_{5.10} = \frac{55}{3}(26a_{1.14} + a_{13.2})$	$a_{5.11} = 286(2.5a_{1.15} - a_{15.1})$			
$a_{69} = 66(a_{2.13} + 6.5a_{14.1})$	$a_{6.10} = \frac{143}{3}(2a_{2.14} + a_{14.2})$				
$a_{79} = -1716a_{1.15} + 1287a_{15.1}$					

biharmonic constraint is to cancel rows or columns. To find out how
note that all rows with dependent coefficients whose i-index is even
arbitrary coefficients whose i-index is also even, and all rows wit
coefficients having odd i-index contain only arbitrary coefficients wh
also odd. Hence one can either cancel *all* rows with even index i, or o
all rows with odd index i, yet the condition $\nabla^4\psi = 0$ remains sati
resulting altered polynomial. If the rows with odd i-index are cancelle
obtain the polynomial valid for the symmetrical model without motio
If in the same polynomial all arbitrary coefficients with j index $= 1$ a
cancelled we obtain a polynomial which yields $v = u = 0$ at $y = 0$; that
function valid for a symmetrical model welded to the base has been o

Returning now to the columns, a parallel situation is noted. Co
dependent coefficients have an even j-index contain only arbitrar
whose j-index is also even; and columns with dependent coefficients
j-index contain only arbitrary coefficients with an odd j-index. Thi
none of the coefficients in columns with an odd j-index are related
coefficients in columns with an even index j, while at least some co
column with, say an even index j are related to some coefficients i
with an even j-index, and the same interrelation holds for columns

	y	y^2	y^3	y^4	y^5	y^6	y^7
11	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
21	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}	a_{28}
31	a_{32}	$a_{33} = -3(a_{15} + a_{51})$	$a_{34} = -5a_{16} - a_{52}$	$a_{35} = -10a_{17} + 5a_{71}$	$a_{36} = -14a_{18} + a_{72}$	$a_{37} = -21a_{19} - 7a_{91}$	$a_{38} = -27a_{1.10} - a_{92}$
41	a_{42}	$a_{43} = -a_{25} - 5a_{61}$	$a_{44} = -\frac{5}{3}(a_{26} + a_{62})$	$a_{45} = -\frac{10}{3}a_{27} + \frac{35}{3}a_{81}$	$a_{46} = -\frac{14}{3}a_{28} + \frac{7}{3}a_{82}$	$a_{47} = -7(a_{29} + 3a_{10.1})$	$a_{48} = -3(3a_{2.10} + a_{10.2})$
51	a_{52}	$a_{53} = 5a_{17} - 10a_{71}$	$a_{54} = \frac{35}{3}a_{18} - \frac{10}{3}a_{72}$	$a_{55} = 35(a_{19} + a_{91})$	$a_{56} = 63a_{1.10} + 7a_{92}$	$a_{57} = 126a_{1.11} - 84a_{11.1}$	$a_{58} = 198a_{1.12} - 12a_{11.2}$
61	a_{62}	$a_{63} = a_{27} - 14a_{81}$	$a_{64} = \frac{7}{3}a_{28} - \frac{14}{3}a_{82}$	$a_{65} = 7a_{29} + 63a_{10.1}$	$a_{66} = \frac{63}{5}(a_{2.10} + a_{10.2})$	$a_{67} = \frac{126}{5}a_{2.11} - \frac{924}{5}a_{12.1}$	$a_{68} = \frac{198}{5}a_{2.12} - \frac{132}{5}a_{12.2}$
71	a_{72}	$a_{73} = -7(a_{19} + 3a_{9.1})$	$a_{74} = -7(3a_{1.10} + a_{9.2})$	$a_{75} = -84a_{1.11} + 126a_{11.1}$	$a_{76} = -\frac{924}{5}a_{1.12} + \frac{126}{5}a_{11.2}$	$a_{77} = -462(a_{1.13} + a_{13.1})$	$a_{78} = -66(13a_{1.14} + a_{13.2})$
81	a_{82}	$a_{83} = -a_{29} - 27a_{10.1}$	$a_{84} = -3(a_{2.10} + 3a_{10.2})$	$a_{85} = -12a_{2.11} + 198a_{12.1}$	$a_{86} = -\frac{132}{5}a_{2.12} + \frac{198}{5}a_{12.2}$	$a_{87} = -66(a_{2.13} + 13a_{14.1})$	$a_{88} = -858/7 (a_{2.14} + a_{14.2})$
91	a_{92}	$a_{93} = 9a_{1.11} - 36a_{11.1}$	$a_{94} = 33a_{1.12} - 12a_{11.2}$	$a_{95} = 165(a_{1.13} + 2a_{13.1})$	$a_{96} = 66(6.5a_{1.14} + a_{13.2})$	$a_{97} = 1287a_{1.15} - 1716a_{15.1}$	
101	$a_{10.2}$	$a_{10.3} = a_{2.11} - 44a_{12.1}$	$a_{10.4} = \frac{11}{3}(a_{2.12} - 4a_{12.2})$	$a_{10.5} = \frac{55}{3}(a_{2.13} + 26a_{14.1})$	$a_{10.6} = \frac{143}{3}(a_{2.14} + 2a_{14.2})$		
111	$a_{11.2}$	$a_{11.3} = -11(a_{1.13} + 5a_{13.1})$	$a_{11.4} = -\frac{55}{3}(2.6a_{1.14} + a_{13.2})$	$a_{11.5} = 286(-a_{1.15} + 2.5a_{15.1})$			
121	$a_{12.2}$	$a_{12.3} = -(a_{2.13} + 65a_{14.1})$	$a_{12.4} = -\frac{13}{3}(a_{2.14} + 5a_{14.2})$				
131	$a_{13.2}$	$a_{13.3} = 13(a_{1.15} - 6a_{15.1})$					
141	$a_{14.2}$						
151							

y^8	y^9	y^{10}	y^{11}	y^{12}	y^{13}	y^{14}
a_{19}	$a_{1.10}$	$a_{1.11}$	$a_{1.12}$	$a_{1.13}$	$a_{1.14}$	$a_{1.15}$
a_{29}	$a_{2.10}$	$a_{2.11}$	$a_{2.12}$	$a_{2.13}$	$a_{2.14}$	
$a_{39} = -36a_{1.11} + 9a_{11.1}$	$a_{3.10} = -44a_{1.12} + a_{11.2}$	$a_{3.11} = -55a_{1.13} - 11a_{13.1}$	$a_{3.12} = -(65a_{1.14} + a_{13.2})$	$a_{3.13} = 13(-6a_{1.15} + a_{15.1})$		
$a_{49} = -12a_{2.11} + 33a_{12.1}$	$a_{4.10} = \frac{11}{3}(-4a_{2.12} + a_{12.2})$	$a_{4.11} = -\frac{55}{3}(a_{2.13} + 2.6a_{14.1})$	$a_{4.12} = -\frac{13}{3}(5a_{2.14} + 10a_{14.2})$			
$a_{59} = 165(2a_{1.13} + a_{13.1})$	$a_{5.10} = \frac{55}{3}(26a_{1.14} + a_{13.2})$	$a_{5.11} = 286(2.5a_{1.15} - a_{15.1})$				
$a_{69} = 66(a_{2.13} + 6.5a_{14.1})$	$a_{6.10} = \frac{143}{3}(2a_{2.14} + a_{14.2})$					
$a_{79} = -1716a_{1.15} + 1287a_{15.1}$						

biharmonic constraint is to cancel rows or columns. To find out how to do this we note that all rows with dependent coefficients whose i-index is even contain only arbitrary coefficients whose i-index is also even, and all rows with dependent coefficients having odd i-index contain only arbitrary coefficients whose i-index is also odd. Hence one can either cancel *all* rows with even index i, or one can cancel *all* rows with odd index i, yet the condition $\nabla^4\psi = 0$ remains satisfied for the resulting altered polynomial. If the rows with odd i-index are cancelled, we in fact obtain the polynomial valid for the symmetrical model without motion at the base. If in the same polynomial all arbitrary coefficients with j index = 1 and 2 are also cancelled we obtain a polynomial which yields $v = u = 0$ at $y = 0$; that is the stream function valid for a symmetrical model welded to the base has been obtained.

Returning now to the columns, a parallel situation is noted. Columns whose dependent coefficients have an even j-index contain only arbitrary coefficients whose j-index is also even; and columns with dependent coefficients with an odd j-index contain only arbitrary coefficients with an odd j-index. This means that none of the coefficients in columns with an odd j-index are related to any of the coefficients in columns with an even index j, while at least some coefficients in a column with, say an even index j are related to some coefficients in all columns with an even j-index, and the same interrelation holds for columns with an odd

index j. It follows from these relationships that the comprehensive polynomial may be altered, and still remain a solution of $\nabla^4\psi = 0$, if either *all* columns with coefficients with an even j-index are cancelled, or *all* columns with coefficients with an odd j-index are cancelled.

Before applying stream function (31) to physical models, it is desirable to comment on certain problems related to stream-function simulation.

THE PROBLEM OF SATISFYING BOUNDARY STRESSES AND STRAINS

It is important to be aware of the difficulty in polynomial stream functions of satisfying completely the surface strains and stresses existing in physical models. In the model described by Ramberg (1981, pp. 207 ff.), for example, it was not possible to satisfy the condition of vanishing shear at the free vertical front face of the collapsing nappe. The shear at the front face could only be made to vanish if all coefficients in the stream function were reduced to zero, obviously a meaningless situation.

Because the polynomial used in Ramberg (1981) was of rather low degree—degree 5 in y and 3 in x—it seems not an unreasonable assumption that the larger number of arbitrary coefficients in a higher degree polynomial would permit the shear at the front face to vanish without reducing all coefficients to zero. This is unfortunately not so, a fact which can be demonstrated by applying the expression:

$$\dot{\gamma}_{xy} = \partial u/\partial y + \partial v/\partial x = -\partial^2\psi/\partial y^2 + \partial^2\psi/\partial x^2 = 0 \qquad (32)$$

at $x = \pm L$, $0 \leqq y \leqq H$ on stream functions of high degree.

In addition to the problem of simulating a stress free front face when using polynomial stream functions, there are limitations when it comes to exact simulation of a stress free top surface of models. It poses no difficulty to *either* apply a condition of vanishing shear stress at the top face, *or* a condition of vanishing normal stress but it seems not always possible to apply both conditions simultaneously in a polynomial stream function, at least not if the degree is low or moderate.

COEFFICIENT DETERMINATION BY THE METHOD OF EXTREMIZING THE RATE OF ENERGY CHANGE

One drawback due to the above noted discrepancy between physical models and their mathematical stream-function images is that the true physical boundary conditions $\dot{\gamma}_{xy} = 0$ and $\sigma_x = 0$ at the front face or the condition of simultaneous vanishing of $\dot{\gamma}_{xy}$ and σ_y at the top surface cannot be used to determine coefficients, and one may wonder how then the coefficients can be determined? Earlier the author applied the principle of extremum rate of dissipation of energy—or the equivalent extremum rate of decline of potential energy—as a means of determining coefficients (Ramberg, 1981, p. 211). Depending upon whether the for velocities are considered constant during the procedure of determining extremum, the latter is either a maximum (at constant forces) or a mi constant velocities).

In the model of a gravitationally spreading nappe not horizontally p driving force is fixed, viz. the body force of gravity, and the potential gra is the sole supply of energy which keeps the system moving and Therefore gravitational spreading will take the path characterized by ma of decline of potential energy, coupled with the side condition that potential energy change balances the rate of dissipation due to viscous st happens is that the principle of maximum rate of decline of potential maximum rate of dissipation if one prefers—controls the values of the in the stream function in such a manner that the corresponding ve assumes a geometry, which—under the prevailing constraints—gives resistance and accordingly maximum rate of collapse and maximum rate ing of the body. (Cf. Gauss' "Principle of Least Constraint": "The n system of interconnected points, interconnected in any way and submi influence takes place so that the constraints on the system are the leas Van Nostrand's Scientific Encyclopedia, 1959, p. 731.)

With the theory of non-equilibrium thermodynamics in mind, e.g. (1970), interesting to consider the change of entropy associated with changes. In Newtonian viscous flow of incompressible fluids, the structure of the fluid before and after flowage is the same, and the en material remains unchanged during flow at constant temperature. The c change which takes place is therefore due to the heat produced by vis divided by the instantaneous temperature. Maximum rate of dissipation accordingly equivalent to maximum rate of production of entropy.

The energy-extremizing method generates enough relationships to determine all coefficients in the polynomial. Examples of this kind of ca found below.

It admittedly gives grounds for worry that the lack of stress at the fr in the physical model cannot be expressed by derivation from a polync function, and one wonders how significant this lack of exact cor between physical- and mathematical model may be.

A crucial aspect of this problem is the condition that energy is a conserved when derived from a stream function as pointed out on p. 2

For the model considered the principle of conservation of energy is eqn. (33):

$$\Delta\dot{E}_{\text{pot}} + \Delta\dot{E}_{\epsilon\gamma} - \Delta\dot{E}_{\sigma\tau} = 0$$

Here the change of potential energy is reckoned positive when in negative when decreasing; the strain energy is always positive and the e

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In the model of a gravitationally spreading nappe not horizontally pushed, the driving force is fixed, viz. the body force of gravity, and the potential gravity energy is the sole supply of energy which keeps the system moving and deforming. Therefore gravitational spreading will take the path characterized by maximum rate of decline of potential energy, coupled with the side condition that the rate of potential energy change balances the rate of dissipation due to viscous strain. What happens is that the principle of maximum rate of decline of potential energy—or maximum rate of dissipation if one prefers—controls the values of the coefficients in the stream function in such a manner that the corresponding velocity field assumes a geometry, which—under the prevailing constraints—gives minimum resistance and accordingly maximum rate of collapse and maximum rate of spreading of the body. (Cf. Gauss' "Principle of Least Constraint": "The motion of a system of interconnected points, interconnected in any way and submitted to any influence takes place so that the constraints on the system are the least possible". Van Nostrand's Scientific Encyclopedia, 1959, p. 731.)

With the theory of non-equilibrium thermodynamics in mind, e.g., Gyarmati (1970), interesting to consider the change of entropy associated with the energy changes. In Newtonian viscous flow of incompressible fluids, the microscopic structure of the fluid before and after flowage is the same, and the entropy of the material remains unchanged during flow at constant temperature. The only entropy change which takes place is therefore due to the heat produced by viscous strain, divided by the instantaneous temperature. Maximum rate of dissipation of energy is accordingly equivalent to maximum rate of production of entropy.

The energy-extremizing method generates enough relationships to allow us to determine all coefficients in the polynomial. Examples of this kind of calculation are found below.

It admittedly gives grounds for worry that the lack of stress at the free front face in the physical model cannot be expressed by derivation from a polynomial stream function, and one wonders how significant this lack of exact correspondence between physical- and mathematical model may be.

A crucial aspect of this problem is the condition that energy is automatically conserved when derived from a stream function as pointed out on p. 208.

For the model considered the principle of conservation of energy is expressed in eqn. (33):

$$\Delta\dot{E}_{\text{pot}} + \Delta\dot{E}_{\epsilon\gamma} - \Delta\dot{E}_{\sigma\tau} = 0 \qquad (33)$$

Here the change of potential energy is reckoned positive when increasing and negative when decreasing; the strain energy is always positive and the energy change

due to stress acting on moving parts of the boundary is positive if energy is introduced into the body and negative if the body imparts energy to the surroundings.

In the procedure for determining the coefficients by maximizing the rate of energy change, the rate of decline of potential energy was equated with the rate of dissipation due to viscous strain, i.e., $\dot{E}_{pot} + \dot{E}_{\epsilon\gamma}$ was put equal to zero, in agreement with the fact that potential energy was the only driving power in the system. From eqn. (33) we see that a consequence of this is that $\dot{E}_{\sigma\tau} = 0$. In other words, the energy change due to the discrepant boundary stresses—i.e., the stresses which cannot be made to vanish when derived from the stream function even though they are zero in the corresponding physical model—vanishes when integrated over the entire boundary. Or expressed differently: the average energy change due to the discrepant boundary stresses is zero. From this crucial information we conclude that the lack of exact correspondence as regards boundary stresses between a physical model and its polynomial stream-function image may only be of marginal consequence for the dynamic behaviour of the mathematical model. Thus the velocity field calculated from a polynomial stream function should approximate quite well the physical model as long as correct input values for viscosity, density and geometric shapes and dimensions are used.

In view of the above comments it is reassuring to find that the validity of the method of maximizing the rate of dissipation—or rate of declining potential energy—during gravitational viscous collapse has been demonstrated experimentally, (Mulugeta and Ramberg, in prep.).

SIMULATION OF A SPREADING COMPOSITE NAPPE

The use of continuity of velocity and stresses at layer interface

Complete coherence at the base and free slip along the base discussed in Ramberg (1981) represents two extreme situations not likely to be found in the real world. In many nappes and thrust sheets motion occurs along the basal plane in contrast to the coherent model, but the motion is clearly not frictionless as it would be in a free slip model. To incorporate restricted basal motion in the model one could apply a reasonable coefficient of friction assuming slide along a rigid surface, but one could also consider motion in the form of concentrated viscous shear in a relatively thin basal layer. We shall treat the latter possibility and in so doing assign a relatively low Newtonian viscosity to the basal layer whose contact to the overlaying complex is coherent and mobile while exhibiting immobile contact to the rigid basement.

The model to consider is indicated in Fig. 2. It is a double-layer complex riding on a rigid horizontal basement. Both layers are homogeneous with respect to density, viscosity and thickness: ρ_1, η_1, and H_1 are valid for the bottom layer, and

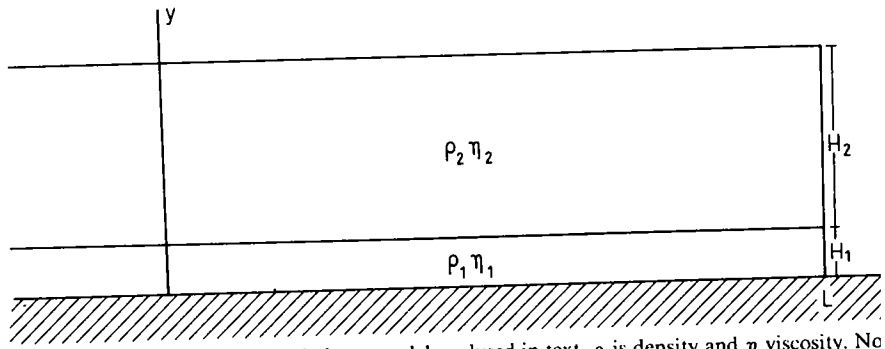

Fig. 2. Right-hand half of double-layer model analysed in text. ρ is density and η viscosity. No symbol ρ is replaced by Ro and η by Mu in the computer-plotted drawings later in this pape

ρ_2, η_2 and H_2 for the top layer. By making H_1 much less than H_2 and η_1 than η_2 the conditions noted above are simulated. However, the theor follows applies to a variety of values of ρ, η, and H. L is the half-lengt symmetric body whose semi-aspect ratios are $R_1 = L/H_1$ and $R_2 = L/H_2$ two layers.

To simulate the model, stream functions of moderately high degree ar Stream function (34), valid for vanishing velocity at the base, is used for layer, and stream function (35) which permits motion both at the upper a boundaries, is applied to the top layer. Both functions are truncated versic comprehensive ψ-function (31). In both functions the rows of terms wit i-index (and even power of x) are excluded, meaning that symmetry exist vertical plane normal to x at $x = 0$ and that the velocity component u in b is zero here.

$$\begin{aligned}\psi_1 &= \left(a'_{23}y_1^2 + a'_{24}y_1^3 + a'_{25}y_1^4 + a'_{26}y_1^5 + a'_{27}y_1^6 + a'_{28}y_1^7\right)x_1 \\ &\quad - \left(a'_{25}y_1^2 + \tfrac{5}{3}a'_{26}y_1^3 + \tfrac{10}{3}a'_{27}y_1^4 + \tfrac{14}{3}a'_{28}y_1^5\right)x_1^3 \\ &\quad + \left(a'_{27}y_1^2 + \tfrac{7}{3}a'_{28}y_1^3\right)x_1^5 \\ &= -\left(ay_1^2 + by_1^3 + cy_1^4 + dy_1^5 + ey_1^6 + fy_1^7\right)x \\ &\quad + \left(cy_1^2 + \tfrac{5}{3}dy_1^3 + \tfrac{10}{3}ey_1^4 + \tfrac{14}{3}fy_1^5\right)x^3 \\ &\quad - \left(ey_1^2 + \tfrac{7}{3}fy_1^3\right)x^5\end{aligned}$$

$$\begin{aligned}\psi_2 &= \left(a_{21} + a_{22}y_2 + a_{23}y_2^2 + a_{24}y_2^3 + a_{25}y_2^4 + a_{26}y_2^5\right)x \\ &\quad + \left[a_{41} + a_{42}y_2 - \left(a_{25} + 5a_{61}\right)y_2^2 - \tfrac{5}{3}\left(a_{26} + a_{62}\right)y_2^3\right]x^3 \\ &\quad + \left(a_{61} + a_{62}y_2\right)x^5\end{aligned}$$

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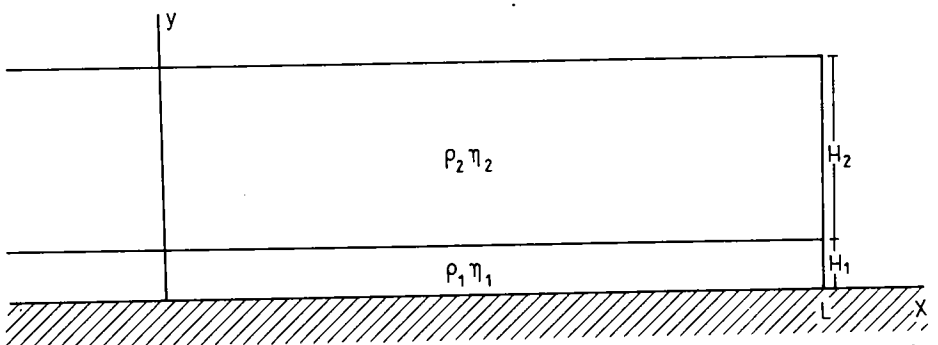


Fig. 2. Right-hand half of double-layer model analysed in text. ρ is density and η viscosity. Note that the symbol ρ is replaced by Ro and η by Mu in the computer-plotted drawings later in this paper.

ρ_2, η_2 and H_2 for the top layer. By making H_1 much less than H_2 and η_1 smaller than η_2 the conditions noted above are simulated. However, the theory which follows applies to a variety of values of ρ, η, and H. L is the half-length of the symmetric body whose semi-aspect ratios are $R_1 = L/H_1$ and $R_2 = L/H_2$ for the two layers.

To simulate the model, stream functions of moderately high degree are chosen. Stream function (34), valid for vanishing velocity at the base, is used for the basal layer, and stream function (35) which permits motion both at the upper and lower boundaries, is applied to the top layer. Both functions are truncated versions of the comprehensive ψ-function (31). In both functions the rows of terms with an odd i-index (and even power of x) are excluded, meaning that symmetry exists about a vertical plane normal to x at $x = 0$ and that the velocity component u in both layers is zero here.

$$\begin{aligned}\psi_1 &= \left(a'_{23}y_1^2 + a'_{24}y_1^3 + a'_{25}y_1^4 + a'_{26}y_1^5 + a'_{27}y_1^6 + a'_{28}y_1^7\right)x_1 \\ &\quad - \left(a'_{25}y_1^2 + \tfrac{5}{3}a'_{26}y_1^3 + \tfrac{10}{3}a'_{27}y_1^4 + \tfrac{14}{3}a'_{28}y_1^5\right)x_1^3 \\ &\quad + \left(a'_{27}y_1^2 + \tfrac{7}{3}a'_{28}y_1^3\right)x_1^5 \\ &= -\left(ay_1^2 + by_1^3 + cy_1^4 + dy_1^5 + ey_1^6 + fy_1^7\right)x \\ &\quad + \left(cy_1^2 + \tfrac{5}{3}dy_1^3 + \tfrac{10}{3}ey_1^4 + \tfrac{14}{3}fy_1^5\right)x^3 \\ &\quad - \left(ey_1^2 + \tfrac{7}{3}fy_1^3\right)x^5 \qquad (34)\end{aligned}$$

$$\begin{aligned}\psi_2 &= \left(a_{21} + a_{22}y_2 + a_{23}y_2^2 + a_{24}y_2^3 + a_{25}y_2^4 + a_{26}y_2^5\right)x \\ &\quad + \left[a_{41} + a_{42}y_2 - (a_{25} + 5a_{61})y_2^2 - \tfrac{5}{3}(a_{26} + a_{62})y_2^3\right]x^3 \\ &\quad + (a_{61} + a_{62}y_2)x^5 \qquad (35)\end{aligned}$$

In ψ_2 and in the first version of ψ_1 the coefficients are indexed exactly as in the comprehensive eqn. (31) in order to show clearly how the truncation and modification of the comprehensive polynomial are performed. We shall keep the original subscripts of the coefficients in ψ_2 but it is practical for the operation that follows that the coefficient notation in ψ_1 is changed as shown in its second version. In ψ_1 the condition $u_1 = v_1 = 0$ at $y = 0$ is accounted for by letting coefficients a_{i1} and a_{i2} vanish, $i = 2$, 4 and 6.

A condition for dynamic equilibrium is that the velocity be continuous across the contact (i.e., $u_1 = u_2$; $v_1 = v_2$ at the contact). A further condition is that the tangential shear stress and the normal stress are continuous at the contact. Together with extremizing the rate of energy dissipation—or rate of potential energy decline—these conditions will be used to determine coefficients.

Equation (34) is valid for $0 \leqq y_1 \leqq H_1$, and eqn. (35) for $0 \leqq y_2 \leqq H_2$. Both equations are valid for $-L \leqq x \leqq +L$.

The following velocities are derived from the stream functions;

$$u_1 = -\frac{\partial \psi_1}{\partial y} = \left(2ay_1 + 3by_1^2 + 4cy_1^3 + 5dy_1^4 + 6ey_1^5 + 7fy_1^6\right)x - \left(2cy_1 + 5dy_1^2 + \tfrac{40}{3}ey_1^3 + \tfrac{70}{3}fy_1^4\right)x^3 + \left(2ey_1 + 7fy_1^2\right)x^5 \tag{36}$$

$$v_1 = \frac{\partial \psi_1}{\partial x} = -\left(ay_1^2 + by_1^3 + cy_1^4 + dy_1^5 + ey_1^6 + fy_1^7\right) + \left(3cy_1^2 + 5dy_1^3 + 10ey_1^4 + 14fy_1^5\right)x^2 - \left(5ey_1^2 + \tfrac{35}{3}fy_1^3\right)x^4 \tag{37}$$

$$u_2 = -\frac{\partial \psi_2}{\partial y} = -\left(a_{22} + 2a_{23}y_2 + 3a_{24}y_2^2 + 4a_{25}y_2^3 + 5a_{26}y_2^4\right)x + \left[-a_{42} + \left(2a_{25} + 10a_{61}\right)y_2 + \left(5a_{26} + 5a_{62}\right)y_2^2\right]x^3 - a_{62}x^5 \tag{38}$$

$$v_2 = \frac{\partial \psi_2}{\partial x} = a_{21} + a_{22}y_2 + a_{23}y_2^2 + a_{24}y_2^3 + a_{25}y_2^4 + a_{26}y_2^5 + \left[3a_{41} + 3a_{42}y_2 - \left(3a_{25} + 15a_{61}\right)y_2^2 - \left(5a_{26} + 5a_{62}\right)y_2^3\right]x^2 + 5\left(a_{61} + a_{62}y_2\right)x^4 \tag{39}$$

At the contact between the two layers $y_1 = H_1$ and $y_2 = 0$. Introduced in eqns. (36) and (38) this leads to:

$$u_1 = \left(2aH_1 + 3bH_1^2 + 4cH_1^3 + 5dH_1^4 + 6eH_1^5 + 7fH_1^6\right)x - \left(2cH_1 + 5dH_1^2 + \tfrac{40}{3}eH_1^3 + \tfrac{70}{3}fH_1^4\right)x^3 + \left(2eH_1 + 7fH_1^2\right)x^5 \tag{40}$$

$$u_2 = -\left(a_{22}x + a_{42}x^3 + a_{62}x^5\right) \tag{41}$$

for the horizontal velocity component. Because u is continuous at the contact for all values of x between $-L$ and $+L$ the sum of terms with the same power of x be equal in the two equations, and we find:

$$a_{62} = -2eH_1 - 7fH_1^2$$

$$a_{42} = 2cH_1 + 5dH_1^2 + \tfrac{40}{3}eH_1^3 + \tfrac{70}{3}fH_1^4$$

$$a_{22} = -2aH_1 - 3bH_1^2 - 4cH_1^3 - 5dH_1^4 - 6eH_1^5 - 7fH_1^6$$

Similarly the condition $v_1 = v_2$ at $y_1 = H_1$ and $y_2 = 0$ yields:

$$a_{61} = -eH_1^2 - \tfrac{7}{3}fH_1^3$$

$$a_{41} = cH_1^2 + \tfrac{5}{3}dH_1^3 + \tfrac{10}{3}eH_1^4 + \tfrac{14}{3}fH_1^5$$

$$a_{21} = -aH_1^2 - bH_1^3 - cH_1^4 - dH_1^5 - eH_1^6 - fH_1^7$$

To apply the condition $\tau_{xy_1} = \tau_{xy_2}$ at the contact we derive the expression τ_{xy} $= \eta(\partial u/\partial y + \partial v/\partial x) = \eta(\partial^2\psi/\partial x^2 - \partial^2\psi/\partial y^2)$ for the two layers:

$$\tau_{xy_1} = \eta_1\dot{\gamma}_{xy_1} = \eta_1\left[\left(2a + 6by_1 + 18cy_1^2 + 30dy_1^3 + 50ey_1^4 + 70fy_1^5\right)x - \left(2c + 10dy_1 + 60ey_1^2 + 140fy_1^3\right)x^3 + \left(2e + 14fy_1\right)x^5\right]$$

$$\tau_{xy_2} = \eta\dot{\gamma}_{xy_2} = \eta_2\left[\left\{-2a_{23} + 6a_{41} - \left(6a_{24} - 6a_{42}\right)y_2 - \left(18a_{25} + 30a_{61}\right)y_2^2 - \left(30a_{26} + 10a_{62}\right)y_2^3\right\}x + \left\{2a_{25} + 30a_{61} + \left(10a_{26} + 30a_{62}\right)y_2\right\}x^3$$

As continuous shear stress is required at all points—at all x—on the cor find after equating τ_{xy_1} at $y_1 = H_1$ with τ_{xy_2} at $y_2 = 0$:

$$f = -\tfrac{1}{7}H_1^{-1}e$$

$$\eta_2\left(2a_{25} + 30a_{61}\right) = -\eta_1\left(2c + 10dH_1 + 40eH_1^2\right)$$

and:

$$\eta_2\left(-2a_{23} + 6a_{41}\right) = \eta_1\left(2a + 6bH_1 + 18cH_1^2 + 30dH_1^3 + 40eH_1^4\right)$$

To make use of the condition $\sigma_{y_1} = \sigma_{y_2}$ at the contact we start with the forr

$$\sigma_y = 2\eta\dot{\epsilon}_y - P$$

and:

$$\mathrm{d}P = \frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y$$

in which $\dot{\epsilon}_y$ and the partial derivatives are derived from the two stream using relations $\dot{\epsilon}_y = \partial v/\partial y = \partial^2\psi/\partial x\partial y$;

$$\partial P/\partial x = \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = -\eta\left(\frac{\partial^3\psi}{\partial x\partial y^2} + \frac{\partial^3\psi}{\partial y^3}\right)$$

and:

$$\partial P/\partial y = \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \rho g = \eta\left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\psi}{\partial x\partial y_2}\right) - \rho g$$

ψ_2 and in the first version of ψ_1 the coefficients are indexed exactly as in the prehensive eqn. (31) in order to show clearly how the truncation and modification of the comprehensive polynomial are performed. We shall keep the original cripts of the coefficients in ψ_2 but it is practical for the operation that follows he coefficient notation in ψ_1 is changed as shown in its second version. In ψ_1 ndition $u_1 = v_1 = 0$ at $y = 0$ is accounted for by letting coefficients a_{i1} and a_{i2} h, $i = 2$, 4 and 6.

condition for dynamic equilibrium is that the velocity be continuous across the ct (i.e., $u_1 = u_2$; $v_1 = v_2$ at the contact). A further condition is that the ntial shear stress and the normal stress are continuous at the contact. Together xtremizing the rate of energy dissipation—or rate of potential energy decline se conditions will be used to determine coefficients.

uation (34) is valid for $0 \leqq y_1 \leqq H_1$, and eqn. (35) for $0 \leqq y_2 \leqq H_2$. Both ons are valid for $-L \leqq x \leqq +L$.

following velocities are derived from the stream functions;

$$\frac{\partial \psi_1}{\partial y} = \left(2ay_1 + 3by_1^2 + 4cy_1^3 + 5dy_1^4 + 6ey_1^5 + 7fy_1^6\right)x - \left(2cy_1 + 5dy_1^2 + \tfrac{40}{3}ey_1^3 + \tfrac{70}{3}fy_1^4\right)x^3 + \left(2ey_1 + 7fy_1^2\right)x^5 \tag{36}$$

$$\frac{\psi_1}{x} = -\left(ay_1^2 + by_1^3 + cy_1^4 + dy_1^5 + ey_1^6 + fy_1^7\right) + \left(3cy_1^2 + 5dy_1^3 + 10ey_1^4 + 14fy_1^5\right)x^2 - \left(5ey_1^2 + \tfrac{35}{3}fy_1^3\right)x^4 \tag{37}$$

$$\frac{\partial \psi_2}{\partial y} = -\left(a_{22} + 2a_{23}y_2 + 3a_{24}y_2^2 + 4a_{25}y_2^3 + 5a_{26}y_2^4\right)x + \left[-a_{42} + (2a_{25} + 10a_{61})y_2 + (5a_{26} + 5a_{62})y_2^2\right]x^3 - a_{62}x^5 \tag{38}$$

$$^2 = a_{21} + a_{22}y_2 + a_{23}y_2^2 + a_{24}y_2^3 + a_{25}y_2^4 + a_{26}y_2^5 + \left[3a_{41} + 3a_{42}y_2 - (3a_{25} + 15a_{61})y_2^2 - (5a_{26} + 5a_{62})y_2^3\right]x^2 + 5(a_{61} + a_{62}y_2)x^4 \tag{39}$$

ontact between the two layers $y_1 = H_1$ and $y_2 = 0$. Introduced in eqns. (36) this leads to:

$$H_1 + 3bH_1^2 + 4cH_1^3 + 5dH_1^4 + 6eH_1^5 + 7fH_1^6\Big)x$$

$$2cH_1 + 5dH_1^2 + \tfrac{40}{3}eH_1^3 + \tfrac{70}{3}fH_1^4\Big)x^3 + \left(2eH_1 + 7fH_1^2\right)x^5 \tag{40}$$

$$_{22}x + a_{42}x^3 + a_{62}x^5\Big) \tag{41}$$

orizontal velocity component. Because u is continuous at the contact for all values of x between $-L$ and $+L$ the sum of terms with the same power of x must be equal in the two equations, and we find:

$$a_{62} = -2eH_1 - 7fH_1^2 \tag{42}$$

$$a_{42} = 2cH_1 + 5dH_1^2 + \tfrac{40}{3}eH_1^3 + \tfrac{70}{3}fH_1^4 \tag{43}$$

$$a_{22} = -2aH_1 - 3bH_1^2 - 4cH_1^3 - 5dH_1^4 - 6eH_1^5 - 7fH_1^6 \tag{44}$$

Similarly the condition $v_1 = v_2$ at $y_1 = H_1$ and $y_2 = 0$ yields:

$$a_{61} = -eH_1^2 - \tfrac{7}{3}fH_1^3 \tag{45}$$

$$a_{41} = cH_1^2 + \tfrac{5}{3}dH_1^3 + \tfrac{10}{3}eH_1^4 + \tfrac{14}{3}fH_1^5 \tag{46}$$

$$a_{21} = -aH_1^2 - bH_1^3 - cH_1^4 - dH_1^5 - eH_1^6 - fH_1^7 \tag{47}$$

To apply the condition $\tau_{xy_1} = \tau_{xy_2}$ at the contact we derive the expression $\tau_{xy} = \eta\dot{\gamma}_{xy}$ $= \eta(\partial u/\partial y + \partial v/\partial x) = \eta(\partial^2\psi/\partial x^2 - \partial^2\psi/\partial y^2)$ for the two layers:

$$\tau_{xy_1} = \eta_1\dot{\gamma}_{xy_1} = \eta_1\big[\left(2a + 6by_1 + 18cy_1^2 + 30dy_1^3 + 50ey_1^4 + 70fy_1^5\right)x - \left(2c + 10dy_1 + 60ey_1^2 + 140fy_1^3\right)x^3 + \left(2e + 14fy_1\right)x^5\big] \tag{48}$$

$$\tau_{xy_2} = \eta\dot{\gamma}_{xy_2} = \eta_2\big[\{-2a_{23} + 6a_{41} - (6a_{24} - 6a_{42})y_2 - (18a_{25} + 30a_{61})y_2^2 - (30a_{26} + 10a_{62})y_2^3\}x + \{2a_{25} + 30a_{61} + (10a_{26} + 30a_{62})y_2\}x^3\big] \tag{49}$$

As continuous shear stress is required at all points—at all x—on the contact we find after equating τ_{xy_1} at $y_1 = H_1$ with τ_{xy_2} at $y_2 = 0$:

$$f = -\tfrac{1}{7}H_1^{-1}e \tag{50}$$

$$\eta_2(2a_{25} + 30a_{61}) = -\eta_1\left(2c + 10dH_1 + 40eH_1^2\right) \tag{51}$$

and:

$$\eta_2(-2a_{23} + 6a_{41}) = \eta_1\left(2a + 6bH_1 + 18cH_1^2 + 30dH_1^3 + 40eH_1^4\right) \tag{52}$$

To make use of the condition $\sigma_{y_1} = \sigma_{y_2}$ at the contact we start with the formulas:

$$\sigma_y = 2\eta\dot{\epsilon}_y - P \tag{53}$$

and:

$$\mathrm{d}P = \frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y \tag{54}$$

in which $\dot{\epsilon}_y$ and the partial derivatives are derived from the two stream functions using relations $\dot{\epsilon}_y = \partial v/\partial y = \partial^2\psi/\partial x\partial y$;

$$\partial P/\partial x = \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = -\eta\left(\frac{\partial^3\psi}{\partial x\partial y^2} + \frac{\partial^3\psi}{\partial y^3}\right) \tag{55}$$

and:

$$\partial P/\partial y = \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \rho g = \eta\left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\psi}{\partial x\partial y_2}\right) - \rho g \tag{56}$$

When applied to stream function ψ_1 this procedure leads to:

$$dP = \eta_1\big[(6b + 12cy_1 + 30dy_1^2 + 40ey_1^3 + 70fy_1^4)x - (10d + 40ey_1 + 140fy_1^2)x^3 + 14fx^5\big]dx + \big[\eta_1\{-2a - 6by_1 - 6cy_1^2 - 10dy_1^3 - 10ey_1^4 - 14fy_1^5 + (6c + 30dy_1 + 60ey_1^2 + 140fy_1^3)x^2 - (10e + 70fy_1)x^4\} - \rho_1 g\big]dy \quad (57)$$

It can be demonstrated that dP in this equation is an exact differential, and integration can be performed accordingly:

$$P = \eta_1\big[(3b + 6cy_1 + 15dy_1^2 + 20ey_1^3 + 35fy_1^4)x^2 - (\tfrac{10}{4}d + 10ey_1 + 35fy_1^2)x^4 + \tfrac{14}{6}fx^6\big] + f(y) \quad (58)$$

Here $f(y)$ is an unknown function which may be found when eqn. (58) is differentiated with respect to y and the resulting derivative is equated with the partial derivative $\partial P/\partial y$ in eqn. (57). Thus:

$$\partial P/\partial y = \eta_1(6c + 30dy_1 + 60ey_1^2 + 140fy_1^3)x^2 - (10e + 70fy_1)x^4 + \partial f(y)/\partial y \quad (59)$$

must equate with the term $\partial P/\partial y$ in eqn. (57), i.e., the quantity in the parenthesis in front of dy. An expression for $\partial f(y)/\partial y$ is accordingly obtained:

$$\partial f(y)/\partial y = \eta_1(-2a - 6by_1 - 6cy_1^2 - 10dy_1^3 - 10ey_1^4 - 14fy_1^5) - \rho_1 g \quad (60)$$

which upon integration yields:

$$f(y) = \eta_1(-2ay_1 - 3by_1^2 - 2cy_1^3 - \tfrac{5}{2}dy_1^4 - 2ey_1^5 - \tfrac{7}{3}fy_1^6) - \rho_1 g y_1 + c_1 \quad (61)$$

Here c_1 is a constant of integration.

$f(y)$ can be introduced in eqn. (58) which in its turn goes into eqn. (53), and the normal stress parallel to y in layer 1 follows when:

$$\dot{\epsilon}_y = \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y}$$

is also inserted as derived from stream function ψ_1.

$$\sigma_{y_1} = 2\eta_1\dot{\epsilon}_{y1} - P_1 = \eta_1(-2ay - 3by_1^2 - 6cy_1^3 - \tfrac{15}{2}dy_1^4 - 10ey_1^5 - \tfrac{35}{3}fy_1^6 + (-3b + 6cy_1 + 15dy_1^2 + 60ey_1^3 + 105fy_1^4)x^2 + (\tfrac{5}{2}d - 10ey_1 - 35fy_1^2)x^4 - \tfrac{7}{3}fx^6) + \rho_1 g y_1 - c_1 \quad (62)$$

A corresponding procedure performed using stream function ψ_2 produces an expression for the normal stress σ_2 in the top layer:

$$\sigma_{y_2} = 2\eta_2\dot{\epsilon}_{y_2} - P_1 = \eta_2\big[2a_{22} + 2a_{23}y_2 - 6a_{41}y_2 + (3a_{24} - 3a_{42})y_2^2 + (6a_{25} + 10a_{62})y_2^3 + (\tfrac{15}{2}a_{26} + \tfrac{5}{2}a_{62})y_2^4 + \{3a_{24} + 9a_{42} - (6a_{25} + 90a_{61})y_2 - (15a_{26} + 45a_{62})y_2^2\}x^2 - (\tfrac{5}{2}a_{26} - \tfrac{25}{2}a_{62})x^4\big] + \rho g y_2 - c_2 \quad (63)$$

Here again c_2 is a constant of integration. Incidentally, neither c_1 nor c_2 are ne for the determination of coefficients in the stream functions, but the values and c_2 are necessary if we wish to know the magnitude of the normal stress i two layers.

σ_{y_1} must equate with σ_{y2} at all x between $-L$ and $+L$ when y_1 is put equ H_1 and $y_2 = 0$ in eqns. (62) and (63). Performing this operation we immediatel that:

$f = 0$

(because $\frac{7}{3}fx^6$ is the only term containing f). A consequence of this is:

$e = a_{62} = a_{61} = 0$ (see eqns. (50), (42) and (45))

Application of these results on the terms multiplied by x^4 in eqns. (62) an leads to:

$a_{26} = -\eta_1/\eta_2 d \equiv -md$ (here $m \equiv \eta_1/\eta_2$)

Equating the collection of terms multiplied by x^2 in the two equations one ob

$\eta_2(3a_{24} + 9a_{42}) = \eta_1(-3b + 6cH_1 + 15dH_1^2)$,

and the relation:

$2\eta_2 a_{22} - c_2 = -\eta_1(2aH_1 + 3bH_1^2 + 6cH_1^3 + \frac{15}{2}dH_1^4) - c_1 + \rho_1 g H_1$

follows when the constant terms are collected and equated

Equation (67) with the expression for a_{42} introduced (eqn. 43) gives:

$a_{24} = -mb + (2m - 6)cH_1 + (5m - 15)dH_1^2$

From eqns. (52), (45) and (65) follows:

$a_{23} = -m\big[a + 3bH_1 + (9 - 3m^{-1})cH_1^2 + (15 - 5m^{-1})dH_1^3\big]$

and from eqns. (51) and (65):

$a_{25} = -m(c + 5dH_1)$

Now all coefficients in stream function ψ_2 are expressed as functions coefficients a to d in stream function ψ_1. The said relationships are:

$a_{21} = -aH_1^2 - bH_1^3 - cH_1^4 - dH_1^5$

$a_{22} = -2aH_1 - 3bH_1^2 - 4cH_1^3 - 5dH_1^4$

$a_{23} = -m\big[a + 3bH_1 + (9 - 3/m)cH_1^2 + (15 - 5/m)dH_1^3\big]$

$a_{24} = -mb + (2m - 6)cH_1 + (5m - 15)dH_1^2$

$a_{25} = -m(c + 5dH_1)$

$a_{26} = -md$

$a_{41} = cH_1^2 + \frac{5}{3}dH_1^3$

$a_{42} = 2cH_1 + 5dH_1^2$

All other coefficients are zero.

en applied to stream function ψ_1 this procedure leads to:

$$= \eta_1\big[(6b + 12cy_1 + 30dy_1^2 + 40ey_1^3 + 70fy_1^4)x - (10\mathrm{d} + 40ey_1 + 140fy_1^2)x^3 + 14fx^5\big]dx + \big[\eta_1\{-2a - 6by_1 - 6cy_1^2 - 10dy_1^3 - 10ey_1^4 - 14fy_1^5 + (6c + 30dy_1 + 60ey_1^2 + 140fy_1^3)x^2 - (10e + 70fy_1)x^4\} - \rho_1 g\big]dy \quad (57)$$

n be demonstrated that dP in this equation is an exact differential, and ration can be performed accordingly:

$$_1\big[(3b + 6cy_1 + 15dy_1^2 + 20ey_1^3 + 35fy_1^4)x^2 - (\tfrac{10}{4}d + 10ey_1 + 35fy_1^2)x^4 + \tfrac{14}{6}fx^6\big] + f(y) \quad (58)$$

$f(y)$ is an unknown function which may be found when eqn. (58) is ntiated with respect to y and the resulting derivative is equated with the l derivative $\partial P/\partial y$ in eqn. (57). Thus:

$$' = \eta_1(6c + 30dy_1 + 60ey_1^2 + 140fy_1^3)x^2 - (10e + 70fy_1)x^4 + \partial f(y)/\partial y \quad (59)$$

quate with the term $\partial \mathrm{P}/\partial \mathrm{y}$ in eqn. (57), i.e., the quantity in the parenthesis in f dy. An expression for $\partial f(y)/\partial y$ is accordingly obtained:

$$/\partial y = \eta_1(-2a - 6by_1 - 6cy_1^2 - 10dy_1^3 - 10ey_1^4 - 14fy_1^5) - \rho_1 g \quad (60)$$

ıpon integration yields:

$$\eta_1(-2ay_1 - 3by_1^2 - 2cy_1^3 - \tfrac{5}{2}dy_1^4 - 2ey_1^5 - \tfrac{7}{3}fy_1^6) - \rho_1 g y_1 + c_1 \quad (61)$$

is a constant of integration.

can be introduced in eqn. (58) which in its turn goes into eqn. (53), and the stress parallel to y in layer 1 follows when:

$$= \frac{\partial^2 \psi}{\partial x \partial y}$$

ıserted as derived from stream function ψ_1.

$$\epsilon_{y1} - P_1 = \eta_1(-2ay - 3by_1^2 - 6cy_1^3 - \tfrac{15}{2}\mathrm{dy}_1^4 - 10\mathrm{ey}_1^5 - \tfrac{35}{3}\mathrm{fy}_1^6 + (-3\mathrm{b} + 6cy_1 + 15dy_1^2 + 60ey_1^3 + 105fy_1^4)x^2 + (\tfrac{5}{2}d - 10ey_1 - 35fy_1^2)x^4 - \tfrac{7}{3}fx^6) + \rho_1 g y_1 - c_1 \quad (62)$$

esponding procedure performed using stream function ψ_2 produces an ı for the normal stress σ_2 in the top layer:

$$_{y_2} - P_1 = \eta_2\big[2a_{22} + 2a_{23}y_2 - 6a_{41}y_2 + (3a_{24} - 3a_{42})y_2^2 + (6\mathrm{a}_{25} + 10\mathrm{a}_{62})y_2^3 + (\tfrac{15}{2}a_{26} + \tfrac{5}{2}a_{62})y_2^4 + \{3a_{24} + 9a_{42} - (6a_{25} + 90a_{61})y_2 - (15a_{26} + 45a_{62})y_2^2\}x^2 - (\tfrac{5}{2}a_{26} - \tfrac{25}{2}a_{62})x^4\big] + \rho \mathrm{g} y_2 - c_2 \quad (63)$$

Here again c_2 is a constant of integration. Incidentally, neither c_1 nor c_2 are needed for the determination of coefficients in the stream functions, but the values of c_1 and c_2 are necessary if we wish to know the magnitude of the normal stress in the two layers.

σ_{y_1} must equate with σ_{y2} at all x between $-L$ and $+L$ when y_1 is put equal to H_1 and $y_2 = 0$ in eqns. (62) and (63). Performing this operation we immediately see that:

$$f = 0 \quad (64)$$

(because $\frac{7}{3}fx^6$ is the only term containing f). A consequence of this is:

$$e = a_{62} = a_{61} = 0 \qquad \text{(see eqns. (50), (42) and (45))} \quad (65)$$

Application of these results on the terms multiplied by x^4 in eqns. (62) and (63) leads to:

$$a_{26} = -\eta_1/\eta_2 d \equiv -md \qquad (\text{here } m \equiv \eta_1/\eta_2) \quad (66)$$

Equating the collection of terms multiplied by x^2 in the two equations one obtains:

$$\eta_2(3a_{24} + 9a_{42}) = \eta_1(-3b + 6cH_1 + 15dH_1^2), \quad (67)$$

and the relation:

$$2\eta_2 a_{22} - c_2 = -\eta_1(2aH_1 + 3bH_1^2 + 6cH_1^3 + \tfrac{15}{2}dH_1^4) - c_1 + \rho_1 g H_1 \quad (68)$$

follows when the constant terms are collected and equated

Equation (67) with the expression for a_{42} introduced (eqn. 43) gives:

$$a_{24} = -mb + (2m - 6)cH_1 + (5m - 15)dH_1^2 \quad (69)$$

From eqns. (52), (45) and (65) follows:

$$a_{23} = -m\big[a + 3bH_1 + (9 - 3m^{-1})cH_1^2 + (15 - 5m^{-1})dH_1^3\big] \quad (70)$$

and from eqns. (51) and (65):

$$a_{25} = -m(c + 5dH_1) \quad (71)$$

Now all coefficients in stream function ψ_2 are expressed as functions of the coefficients a to d in stream function ψ_1. The said relationships are:

$$a_{21} = -aH_1^2 - bH_1^3 - cH_1^4 - dH_1^5 \quad (72)$$

$$a_{22} = -2aH_1 - 3bH_1^2 - 4cH_1^3 - 5dH_1^4 \quad (73)$$

$$a_{23} = -m\big[a + 3bH_1 + (9 - 3/m)cH_1^2 + (15 - 5/m)dH_1^3\big] \quad (74)$$

$$a_{24} = -mb + (2m - 6)cH_1 + (5m - 15)dH_1^2 \quad (75)$$

$$a_{25} = -m(c + 5dH_1) \quad (76)$$

$$a_{26} = -md \quad (77)$$

$$a_{41} = cH_1^2 + \tfrac{5}{3}dH_1^3 \quad (78)$$

$$a_{42} = 2cH_1 + 5dH_1^2 \quad (79)$$

All other coefficients are zero.

In order to reduce the number of unknown coefficients further, we use the condition of vanishing shear stress parallel to the surface at $y_2 = H_2$. Equation (49) with $f = e = a_{62} = a_{61} = 0$ and $y_2 = H_2$ introduced takes the form:

$$\tau_{XY_2} = \eta_2\left[\left(-2a_{23} + 6a_{41} - 6(a_{24} - a_{42})H_2 - 18a_{25}H^2 - 30a_{26}H_2^3\right)x + (2a_{25} + 10a_{26}H_2)x^3\right] = 0 \qquad (80)$$

When in this equation a_{23}, a_{24} etc. are transformed to a, b, c, and d by means of the formulas on p. 223, and the terms in the two parentheses in front of x^3 and x are separately equated to zero, c and d follow as functions of a and b:

$$c = \left[a + (3 + 3/h)H_1 b\right]/DH_1^2 = \frac{\alpha_1}{H_1^2}a + \frac{\beta_1}{H_1}b \qquad (81)$$

and:

$$d = -\left[a + (3 + 3/h)H_1 b\right]/\left[D(5 + 5/h)H_1^3\right] = \frac{\alpha_2}{H_1^3}a + \frac{\beta_2}{H_1^2}b \qquad (82)$$

Here:

$$D = 6/h - 9 - 9/h^2 - 24/hm + \left(3 + 3/h^3 - 3/h + 9/h^2 + 12/hm\right)/(1 + 1/h),$$

$$h = H_1/H_2, \quad m = \eta_1/\eta_2; \quad \alpha_1 = 1/D; \quad \beta_1 = (3 + 3/h)/D;$$

$$\alpha_2 = -1/D(5 + 5/h); \quad \beta_2 = -(3 + 3/h)/D(5 + 5/h)$$

These formulas, combined with the expressions for a_{ij} as functions of a, b, c, and d, transform all coefficients in the stream functions and their derivatives to a and b. Hence we end up with only two unknown coefficients, a and b, and these will be determined by the method of energy extremization. To use this method, expressions for the strain-energy rate as well as the rate of change of potential energy are needed. Unfortunately the energies require quite lengthy expressions and these are given in full in Appendix A. Here only a brief summary of the procedure for determining the energies will be given.

The instantaneous rate of change of strain energy is obtained by integrating the "specific" strain energy rate (i.e., the energy rate per unit volume),

$$4\eta_i\dot{\epsilon}_x^2 = 4\eta_i\left(-\frac{\partial^2\psi}{\partial y\partial x}\right)^2 \quad \text{and} \quad \eta_i\dot{\gamma}_{xy}^2 = \eta_i\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)^2$$

over a vertical slice of unit thickness parallel to the xy plane across the model, using η_1 and ψ_1 for the part of the slice which cuts layer 1 and η_2 and ψ_2 for the part which cuts layer 2. This leads to equations (A1) to (A4) in Appendix A. When in these equations the coefficients, c, d, A_{22}, A_{23} etc are transformed to a and the equations on p. 223, we find:

$$\dot{E}_{\epsilon\gamma 1} = \eta_1\left(A_{\epsilon\gamma_1}H_2^4a^2 + B_{\epsilon\gamma_1}H_2^6b^2 + C_{\epsilon\gamma_1}H_2^5ab\right)$$

and:

$$\dot{E}_{\epsilon\gamma 2} = \eta_2\left(A_{\epsilon\gamma_2}H_2^4a^2 + B_{\epsilon\gamma_2}H_2^6b^2 + C_{\epsilon\gamma_2}H_2^5ab\right)$$

for the instantaneous strain-energy rates referring to the portions of the sl cuts through layer 1 and layer 2, respectively. Here the coefficients $A_{\epsilon\gamma_i}$, $C_{\epsilon\gamma_i}$, $i = 1, 2$, are quite involved functions of material and geometric pro the two layers, much too lengthy to present in full here. The interested referred to the computer program, Appendix B, in which the symb corresponds to $A_{\epsilon\gamma_1}$, *Estrb1* to $B_{\epsilon\gamma_1}$, *Estrab1* to $C_{\epsilon\gamma_1}$, *Estra2* to $A_{\epsilon\gamma_2}$, *Estr* and finally *Estrab2* corresponds to $C_{\epsilon\gamma_2}$. Note that H_2 is also used in the for layer 1; here the transformation $H_1 \to H_2$ is "hidden" in the coeffici $B_{\epsilon\gamma_1}$, and $C_{\epsilon\gamma_1}$.

The rate of change of potential energy for the slice across the model is integrating the "specific" potential energy rate, $\rho_i g v_i$ over the cross-cut using ρ_1 and v_1 for the part which cuts layer 1 and ρ_2 and v_2 for the pa through layer 2; see eqns. (A5) and (A6) in Appendix A and see also p. transforming c, d, a_{22}, a_{23} etc to a and b by use of the equations on p instantaneous potential energy rate is expressed as follows:

$$\dot{E}_{\text{pot}_1} = \rho_1 g\left(A_{p_1}H_2^4a + B_{p_1}H_2^5b\right)$$

$$\dot{E}_{\text{pot}_2} = \rho_2 g\left(A_{p_2}H_2^4a + B_{p_2}H_2^5b\right)$$

Here the transformation $H_1 \to H_2$ in eqn. (85) is concealed in the coeff and B_{p_1}. The somewhat involved terms of the coefficients A_{p_i} and B_{p_i}, i found in the computer program, Appendix B, where *Epota1*, *Epotb1*, *E Epotb2* correspond to A_{p_1}, B_{p_1}, A_{p_2} and B_{p_2}, respectively.

A condition to consider when using the rate of energy change for d coefficients is that the only energy available for deforming and moving model is gravitational energy. Hence the theorem of conservation of ener that the rate of decline of potential energy equals the rate of dissipat viscous resistance, and the Gaussian principle of least constraint requi rate of decline of potential energy shall assume maximal value under th boundary conditions. Since potential energy change equals energy dissip strain we may also say that the rate of dissipation of energy by viscou assume maximal value.

The problem is accordingly to maximize the potential energy rate t function of the unknown coefficients a and b, and/or to maximize the s

use of vanishing shear strain at top face and energy extremization

order to reduce the number of unknown coefficients further, we use the ition of vanishing shear stress parallel to the surface at $y_2 = H_2$. Equation (49) $f = e = a_{62} = a_{61} = 0$ and $y_2 = H_2$ introduced takes the form:

$$= \eta_2\left[\left(-2a_{23} + 6a_{41} - 6(a_{24} - a_{42})H_2 - 18a_{25}H^2 - 30a_{26}H_2^3\right)x + (2a_{25} + 10a_{26}H_2)x^3\right] = 0 \qquad (80)$$

in this equation a_{23}, a_{24} etc. are transformed to a, b, c, and d by means of rmulas on p. 223, and the terms in the two parentheses in front of x^3 and x parately equated to zero, c and d follow as functions of a and b:

$$+ (3 + 3/h)H_1 b]/DH_1^2 = \frac{\alpha_1}{H_1^2}a + \frac{\beta_1}{H_1}b \qquad (81)$$

$$a + (3 + 3/h)H_1 b]/[D(5 + 5/h)H_1^3] = \frac{\alpha_2}{H_1^3}a + \frac{\beta_2}{H_1^2}b \qquad (82)$$

$h - 9 - 9/h^2 - 24/hm + (3 + 3/h^3 - 3/h + 9/h^2 + 12/hm)/(1 + 1/h)$,

$/H_2$, $m = \eta_1/\eta_2$; $\alpha_1 = 1/D$; $\beta_1 = (3 + 3/h)/D$;

$1/D(5 + 5/h)$; $\beta_2 = -(3 + 3/h)/D(5 + 5/h)$

ormulas, combined with the expressions for a_{ij} as functions of a, b, c, and form all coefficients in the stream functions and their derivatives to a and b. ve end up with only two unknown coefficients, a and b, and these will be ıed by the method of energy extremization. To use this method, expressions strain-energy rate as well as the rate of change of potential energy are Unfortunately the energies require quite lengthy expressions and these are full in Appendix A. Here only a brief summary of the procedure for ing the energies will be given.

ıstantaneous rate of change of strain energy is obtained by integrating the " strain energy rate (i.e., the energy rate per unit volume),

$$\eta_i\left(-\frac{\partial^2\psi}{\partial y\partial x}\right)^2 \quad \text{and} \quad \eta_i\dot{\gamma}_{x,y}^2 = \eta_i\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)^2$$

tical slice of unit thickness parallel to the xy plane across the model, using ı for the part of the slice which cuts layer 1 and η_2 and ψ_2 for the part s layer 2. This leads to equations (A1) to (A4) in Appendix A. When in these equations the coefficients, c, d, A_{22}, A_{23} etc are transformed to a and b, using the equations on p. 223, we find:

$$\dot{E}_{\epsilon\gamma 1} = \eta_1\left(A_{\epsilon\gamma_1}H_2^4a^2 + B_{\epsilon\gamma_1}H_2^6b^2 + C_{\epsilon\gamma_1}H_2^5ab\right) \qquad (83)$$

and:

$$\dot{E}_{\epsilon\gamma 2} = \eta_2\left(A_{\epsilon\gamma_2}H_2^4a^2 + B_{\epsilon\gamma_2}H_2^6b^2 + C_{\epsilon\gamma_2}H_2^5ab\right) \qquad (84)$$

for the instantaneous strain-energy rates referring to the portions of the slice which cuts through layer 1 and layer 2, respectively. Here the coefficients $A_{\epsilon\gamma_i}$, $B_{\epsilon\gamma_i}$ and $C_{\epsilon\gamma_i}$, $i = 1, 2$, are quite involved functions of material and geometric properties of the two layers, much too lengthy to present in full here. The interested reader is referred to the computer program, Appendix B, in which the symbol *Estra1* corresponds to $A_{\epsilon\gamma_1}$, *Estrb1* to $B_{\epsilon\gamma_1}$, *Estrab1* to $C_{\epsilon\gamma_1}$, *Estra2* to $A_{\epsilon\gamma_2}$, *Estrb2* to $B_{\epsilon\gamma_2}$ and finally *Estrab2* corresponds to $C_{\epsilon\gamma_2}$. Note that H_2 is also used in the equation for layer 1; here the transformation $H_1 \to H_2$ is "hidden" in the coefficients $A_{\epsilon\gamma_1}$, $B_{\epsilon\gamma_1}$, and $C_{\epsilon\gamma_1}$.

The rate of change of potential energy for the slice across the model is found by integrating the "specific" potential energy rate, $\rho_i g v_i$ over the cross-cutting slice, using ρ_1 and v_1 for the part which cuts layer 1 and ρ_2 and v_2 for the part cutting through layer 2; see eqns. (A5) and (A6) in Appendix A and see also p. 239. After transforming c, d, a_{22}, a_{23} etc to a and b by use of the equations on p. 223, the instantaneous potential energy rate is expressed as follows:

$$\dot{E}_{\text{pot}_1} = \rho_1 g\left(A_{p_1}H_2^4a + B_{p_1}H_2^5b\right) \qquad (85)$$

$$\dot{E}_{\text{pot}_2} = \rho_2 g\left(A_{p_2}H_2^4a + B_{p_2}H_2^5b\right) \qquad (86)$$

Here the transformation $H_1 \to H_2$ in eqn. (85) is concealed in the coefficients A_{p_1} and B_{p_1}. The somewhat involved terms of the coefficients A_{p_i} and B_{p_i}, $i = 1, 2$, are found in the computer program, Appendix B, where *Epota1*, *Epotb1*, *Epota2* and *Epotb2* correspond to A_{p_1}, B_{p_1}, A_{p_2} and B_{p_2}, respectively.

A condition to consider when using the rate of energy change for determining coefficients is that the only energy available for deforming and moving the present model is gravitational energy. Hence the theorem of conservation of energy requires that the rate of decline of potential energy equals the rate of dissipation due to viscous resistance, and the Gaussian principle of least constraint requires that the rate of decline of potential energy shall assume maximal value under the prevailing boundary conditions. Since potential energy change equals energy dissipation due to strain we may also say that the rate of dissipation of energy by viscous strain will assume maximal value.

The problem is accordingly to maximize the potential energy rate treated as a function of the unknown coefficients a and b, and/or to maximize the strain energy

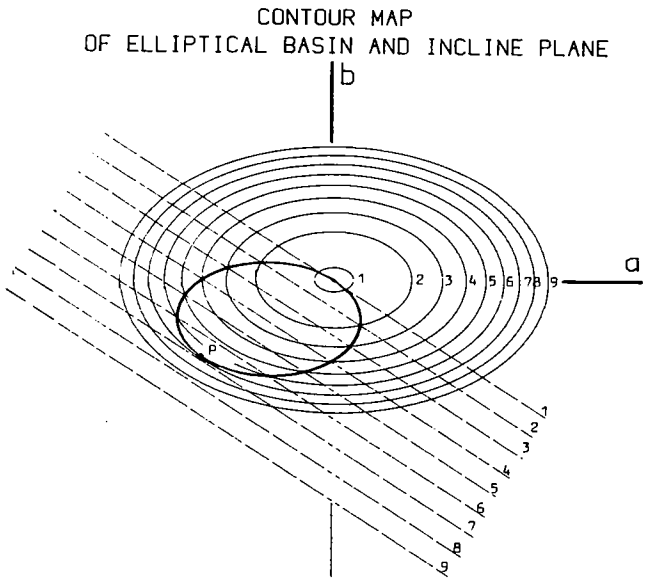


Fig. 3. Contours representing energy increasing from arbitrary values 1 to 9. Solid closed curve passing through origin and the point P marks the intersection between the basin and the plane, P coinciding with maximum energy on the intersection curve. The coordinates for the point P are the sought values for the coefficients a and b. Example showing general principle.

rate also treated as a function of a and b subject to the side-condition that $\dot{E}_{\text{pot}} = \dot{E}_{\epsilon\gamma}$. The situation may be visualized (Fig. 3) when one realizes that $\dot{E}_{\epsilon\gamma} = \dot{E}_{\epsilon\gamma_1} + \dot{E}_{\epsilon\gamma_2}$ describes a quadric surface in the space defined by the axes $\dot{E}_{\epsilon\gamma}$, a and b; and $\dot{E}_{\text{pot}} = \dot{E}_{\text{pot}_1} + \dot{E}_{\text{pot}_2}$ describes a plane in the same space, the axes for $\dot{E}_{\epsilon\gamma}$ and $\dot{E}_{\text{pot}}$ coinciding. Assuming that the $\dot{E}_{\epsilon\gamma}$- or $\dot{E}_{\text{pot}}$ axis is vertical and the a- and b axes are horizontal one finds that $\dot{E}_{\epsilon\gamma} = f_{\epsilon\gamma}(a,b)$ defines a basin whose lowest point coincides with the origin of the coordinate system, and whose elliptical horizontal cross-section area increases with height on the $\dot{E}$-axis. The plane $\dot{E}_{\text{pot}} = f_{\text{pot}}(a,b)$ is generally inclined to the axes and contains the origin. The side condition $\dot{E}_{\epsilon\gamma} + \dot{E}_{\text{pot}} = 0$ defines the intersection curve between the two surfaces. It is the maximal $\dot{E}$-value on this intersection we seek, and especially the a- and b values which correspond to maximum energy rate. As demonstrated in Fig. 3 the extreme values coincide with the condition $(\partial a/\partial b)_{\epsilon\gamma} = (\partial a/\partial b)_{\text{pot}}$. Here subscripts $\epsilon\gamma$ and pot refer to the partial derivative $\partial a/\partial b$ of the strain energy function and of the potential energy function, respectively.

It is easy to see that one extremum—the minimum—coincides with $\dot{E} = 0$, $a = b = 0$, but the maximum depends upon the inclination of the plane $\dot{E}_{\text{pot}} = f_{\text{pot}}(a,b)$ with respect to the $\dot{E}$-axis, and upon the exact shape of the surface $\dot{E}_{\epsilon\gamma} = f_{\epsilon\gamma}(a,b)$.

It is convenient to put the energy equations:

$$\dot{E}_{\epsilon\gamma} = \dot{E}_{\epsilon\gamma_1} + \dot{E}_{\epsilon\gamma_2} = \eta_2\left[\left(\eta_{1:2}A_{\epsilon\gamma_1} + A_{\epsilon\gamma_2}\right)a^2H_2^4 + \left(\eta_{1:2}B_{\epsilon\gamma_1} + B_{\epsilon\gamma_2}\right)b^2H_2^6 + \left(\eta_{1:2}C_{\epsilon\gamma_1} + C_{\epsilon\gamma_2}\right)abH_2^5\right]$$

and:

$$\dot{E}_{\text{pot}} = \dot{E}_{\text{pot}_1} + \dot{E}_{\text{pot}_2} = \rho_2 g\left[\left(\rho_{1:2}A_{p_1} + A_{p_2}\right)aH_2^4 + \left(\rho_{1:2}B_{p_1} + B_{p_2}\right)bH_2^5\right]$$

in the simpler identical forms below:

$$\dot{E}_{\epsilon\gamma} = \eta_2\left(A_{\epsilon\gamma}a_2^2H_2^4 + B_{\epsilon\gamma}b^2H_2^6 + C_{\epsilon\gamma}abH_2^5\right)$$

and:

$$\dot{E}_{\text{pot}} = \rho_2 g\left(A_{\text{pot}a}H_2^4 + B_{\text{pot}b}\mathrm{H}_2^5\right)$$

Here the new coefficients $A_{\epsilon\gamma}$, $B_{\epsilon\gamma}$, $C_{\epsilon\gamma}$, A_{pot} and B_{pot} follow from the id eqns. (87) and (89) and of eqns. (88) and (90), respectively.

By implicit differentiation we find:

$$\left.\frac{\partial a}{\partial b}\right]_{\epsilon\gamma} = -\frac{2B_{\epsilon\gamma}bH_2^2 + C_{\epsilon\gamma}aH_2}{2A_{\epsilon\gamma}a + C_{\epsilon\gamma}bH_2}$$

$$\left.\frac{\partial a}{\partial b}\right]_{\text{pot}} = -\frac{B_{\text{pot}}H_2}{A_{\text{pot}}}$$

When $\left.\frac{\partial a}{\partial b}\right]_{\epsilon\gamma}$ is put equal to $\left.\frac{\partial a}{\partial b}\right]_{\text{pot}}$, a and b become related:

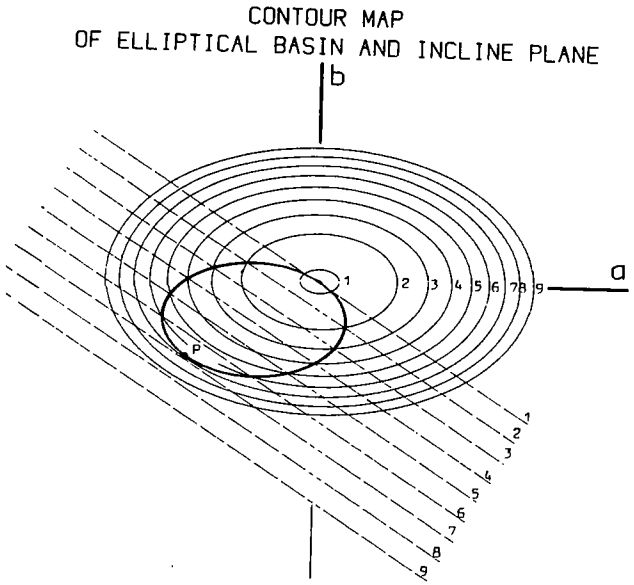
$$b = \frac{2A_{\epsilon\gamma}\frac{B_{\text{pot}}}{A_{\text{pot}}} - C_{\epsilon\gamma}}{2B_{\epsilon\gamma}H_2 - \frac{B_{\text{pot}}}{A_{\text{pot}}}C_{\epsilon\gamma}H_2}a \equiv \pi a$$

The use of this relationship in connection with the condition $\dot{E}_{\epsilon\gamma} + \dot{E}_{\text{pot}} =$ two values of a, namely $a = 0$ and:

$$a = -\frac{\rho_2 g\left(A_{\text{pot}} + B_{\text{pot}}\pi H_2\right)}{\eta_2\left(A_{\epsilon\gamma} + B_{\epsilon\gamma}\pi^2H_2^2 + C\pi H_2\right)}$$

The corresponding values of b follow when a is introduced in eqn. (93) used as a symbol for the fraction in front of a, not as a symbol for 3.14

Now all coefficients in the two stream functions (34) and (35) for th known functions of material and geometric parameters of the two la model. The pertinent material parameters are viscosities and densiti relevant geometric parameters are thicknesses and aspect ratios. This me



3. Contours representing energy increasing from arbitrary values 1 to 9. Solid closed curve passing igh origin and the point P marks the intersection between the basin and the plane, P coinciding maximum energy on the intersection curve. The coordinates for the point P are the sought values ıe coefficients a and b. Example showing general principle.

also treated as a function of a and b subject to the side-condition that $= \dot{E}_{\epsilon\gamma}$. The situation may be visualized (Fig. 3) when one realizes that $\dot{E}_{\epsilon\gamma} = \dot{E}_{\epsilon\gamma_1}$ describes a quadric surface in the space defined by the axes $\dot{E}_{\epsilon\gamma}$, a and b; $\dot{E}_{\text{pot}} = \dot{E}_{\text{pot}_1} + \dot{E}_{\text{pot}_2}$ describes a plane in the same space, the axes for $\dot{E}_{\epsilon\gamma}$ and coinciding. Assuming that the $\dot{E}_{\epsilon\gamma}$- or $\dot{E}_{\text{pot}}$ axis is vertical and the a- and b axes ıorizontal one finds that $\dot{E}_{\epsilon\gamma} = f_{\epsilon\gamma}(a,b)$ defines a basin whose lowest point ides with the origin of the coordinate system, and whose elliptical horizontal ·section area increases with height on the $\dot{E}$-axis. The plane $\dot{E}_{\text{pot}} = f_{\text{pot}}(a,b)$ is ally inclined to the axes and contains the origin. The side condition $\dot{E}_{\epsilon\gamma} + \dot{E}_{\text{pot}}$ lefines the intersection curve between the two surfaces. It is the maximal ıe on this intersection we seek, and especially the a- and b values which pond to maximum energy rate. As demonstrated in Fig. 3 the extreme values de with the condition $(\partial a/\partial b)_{\epsilon\gamma} = (\partial a/\partial b)_{\text{pot}}$. Here subscripts $\epsilon\gamma$ and pot to the partial derivative $\partial a/\partial b$ of the strain energy function and of the ial energy function, respectively.

s easy to see that one extremum—the minimum—coincides with $\dot{E} = 0$, = 0, but the maximum depends upon the inclination of the plane $\dot{E}_{\text{pot}} = b$) with respect to the $\dot{E}$-axis, and upon the exact shape of the surface $_{\epsilon\gamma}(a,b)$.

It is convenient to put the energy equations:

$$\dot{E}_{\epsilon\gamma} = \dot{E}_{\epsilon\gamma_1} + \dot{E}_{\epsilon\gamma_2} = \eta_2\left[\left(\eta_{1:2}A_{\epsilon\gamma_1} + A_{\epsilon\gamma_2}\right)a^2H_2^4 + \left(\eta_{1:2}B_{\epsilon\gamma_1} + B_{\epsilon\gamma_2}\right)b^2H_2^6 + \left(\eta_{1:2}C_{\epsilon\gamma_1} + C_{\epsilon\gamma_2}\right)abH_2^5\right] \tag{87}$$

and:

$$\dot{E}_{\text{pot}} = \dot{E}_{\text{pot}_1} + \dot{E}_{\text{pot}_2} = \rho_2 g\left[\left(\rho_{1:2}A_{p_1} + A_{p_2}\right)aH_2^4 + \left(\rho_{1:2}B_{p_1} + B_{p_2}\right)bH_2^5\right] \tag{88}$$

in the simpler identical forms below:

$$\dot{E}_{\epsilon\gamma} = \eta_2\left(A_{\epsilon\gamma}a_2^2H_2^4 + B_{\epsilon\gamma}b^2H_2^6 + C_{\epsilon\gamma}abH_2^5\right) \tag{89}$$

and:

$$\dot{E}_{\text{pot}} = \rho_2 g\left(A_{\text{pot}a}H_2^4 + B_{\text{pot}b}H_2^5\right) \tag{90}$$

Here the new coefficients $A_{\epsilon\gamma}$, $B_{\epsilon\gamma}$, $C_{\epsilon\gamma}$, A_{pot} and B_{pot} follow from the identity of eqns. (87) and (89) and of eqns. (88) and (90), respectively.

By implicit differentiation we find:

$$\left.\frac{\partial a}{\partial b}\right]_{\epsilon\gamma} = -\frac{2B_{\epsilon\gamma}bH_2^2 + C_{\epsilon\gamma}aH_2}{2A_{\epsilon\gamma}a + C_{\epsilon\gamma}bH_2} \tag{91}$$

$$\left.\frac{\partial a}{\partial b}\right]_{\text{pot}} = -\frac{B_{\text{pot}}H_2}{A_{\text{pot}}} \tag{92}$$

When $\left.\frac{\partial a}{\partial b}\right]_{\epsilon\gamma}$ is put equal to $\left.\frac{\partial a}{\partial b}\right]_{\text{pot}}$, a and b become related:

$$b = \frac{2A_{\epsilon\gamma}\frac{B_{\text{pot}}}{A_{\text{pot}}} - C_{\epsilon\gamma}}{2B_{\epsilon\gamma}H_2 - \frac{B_{\text{pot}}}{A_{\text{pot}}}C_{\epsilon\gamma}H_2}a \equiv \pi a \tag{93}$$

The use of this relationship in connection with the condition $\dot{E}_{\epsilon\gamma} + \dot{E}_{\text{pot}} = 0$ leads to two values of a, namely $a = 0$ and:

$$a = -\frac{\rho_2 g\left(A_{\text{pot}} + B_{\text{pot}}\pi H_2\right)}{\eta_2\left(A_{\epsilon\gamma} + B_{\epsilon\gamma}\pi^2H_2^2 + C\pi H_2\right)} \tag{94}$$

The corresponding values of b follow when a is introduced in eqn. (93). (π is here used as a symbol for the fraction in front of a, not as a symbol for 3.14.)

Now all coefficients in the two stream functions (34) and (35) for the model are known functions of material and geometric parameters of the two layers in the model. The pertinent material parameters are viscosities and densities and the relevant geometric parameters are thicknesses and aspect ratios. This means that the

TABLE 1

Velocity u in cm/year at different levels at the right hand vertical boundary of model. $R = 0.5$, $\eta_2 = \eta_1 = 10^{22}$ P, $\rho = 2.8$ g/cm^3, $H = 5000$ m, N is height in parts of H (see Figs. 4B, 5 and 6A)

u_2	N	u_2	N
−0.0353177800	1	0.1516899015	0.4
0.0186560141	0.9	0.1426623379	0.3
0.0645197947	0.8	0.1174261573	0.2
0.1017673992	0.7	0.0734505460	0.1
0.1295552230	0.6	0.0078672487	0
0.1467022198	0.5		

instantaneous velocity field and stress field of the model are known through eqns. (36) to (39) and eqns. (48), (49), (62) and (63) when the relevant viscosities, densities, thicknesses and aspect ratios are put in the coefficients. The instantaneous rate of decline of potential energy as well as the rate of dissipation also follow automatically via eqns. (A2) to (A6), Appendix A.

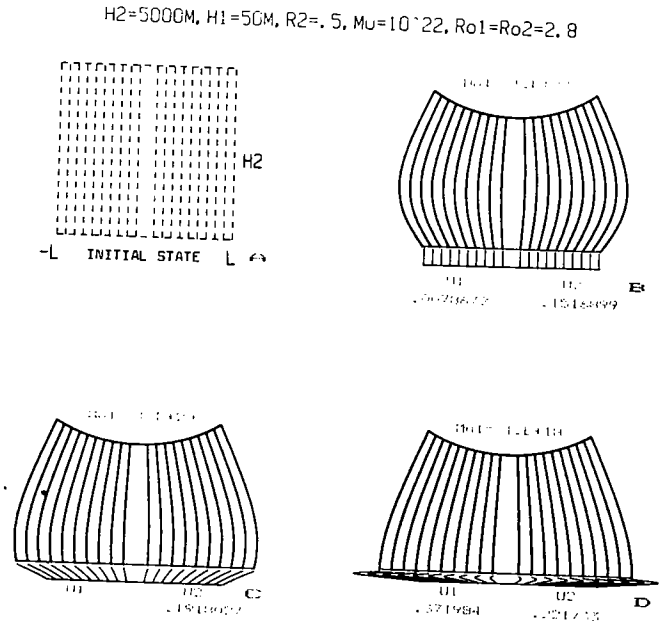


Fig. 4. Deformation of double-layer model with aspect ratio $R_2 = 0.5$ for layer 2. In B, C and D height of layer 1 is exaggerated 10-fold to show the deformation of initially straight vertical markers. As the computer does not print Greek symbols the viscosity has been given the symbol *Mu* and density *Ro*, viscosity has been given in poises (P) and density in g/cm^3. Velocities in cm/yr refer to maximal values at the right-hand initially vertical face. See also Table 1.

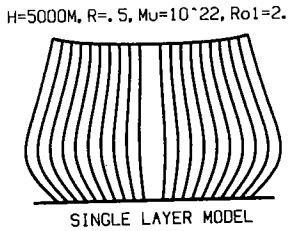


Fig. 5. Deformation of single-layer model after creep with unchanged initial velocity during 5·1(Compare experiment, Fig. 6A. Based on stream function (34) but with 2 orders higher degree, .

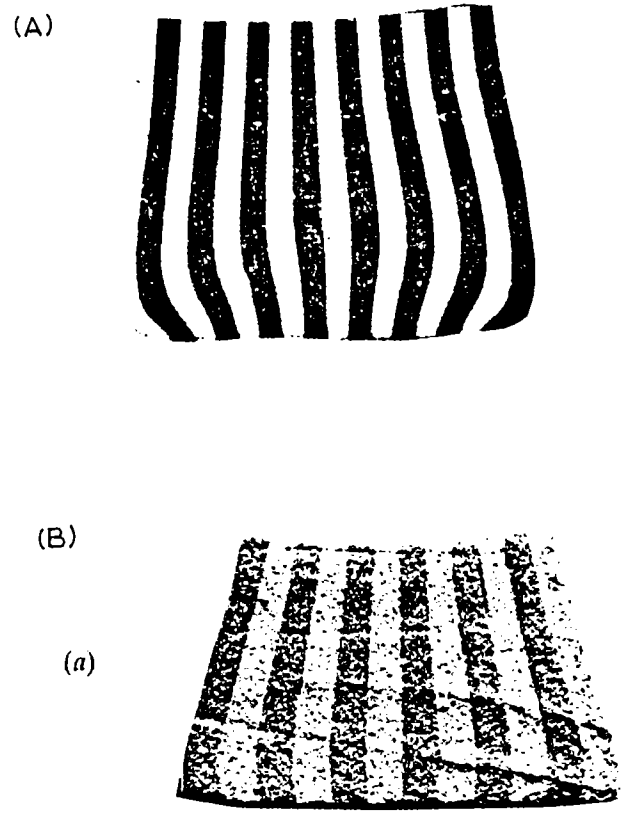


Fig. 6. A. Spreading under own weight of rectangular parallelpiped of viscous silicone put Coherence at base. Early stage of deformation shown. Note similarity between early sta stream-function simulation, Fig. 5. B. Spreading under own weight of rectangular parallelpipe silicone putty. $R = 0.5$. Free slip at base (silicone body floating on mercury). Note similarit 4D.

BLE 1

locity u in cm/year at different levels at the right hand vertical boundary of model. $R = 0.5$, $= \eta_1 = 10^{22}$ P, $\rho = 2.8$ g/cm^3, $H = 5000$ m, N is height in parts of H (see Figs. 4B, 5 and 6A)

	N	u_2	N
.0353177800	1	0.1516899015	0.4
86560141	0.9	0.1426623379	0.3
45197947	0.8	0.1174261573	0.2
17673992	0.7	0.0734505460	0.1
95552230	0.6	0.0078672487	0
67022198	0.5		

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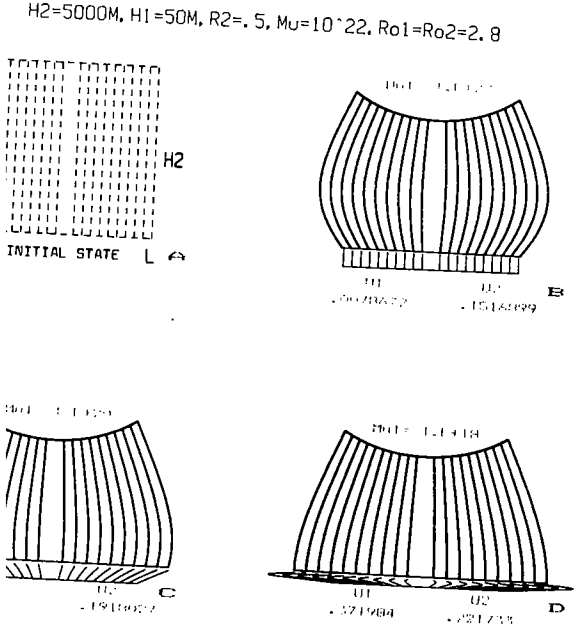

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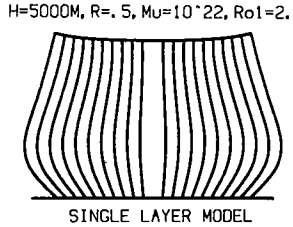

Fig. 5. Deformation of single-layer model after creep with unchanged initial velocity during $5 \cdot 10^5$ years. Compare experiment, Fig. 6A. Based on stream function (34) but with 2 orders higher degree, $x^7 \cdot y^9$.

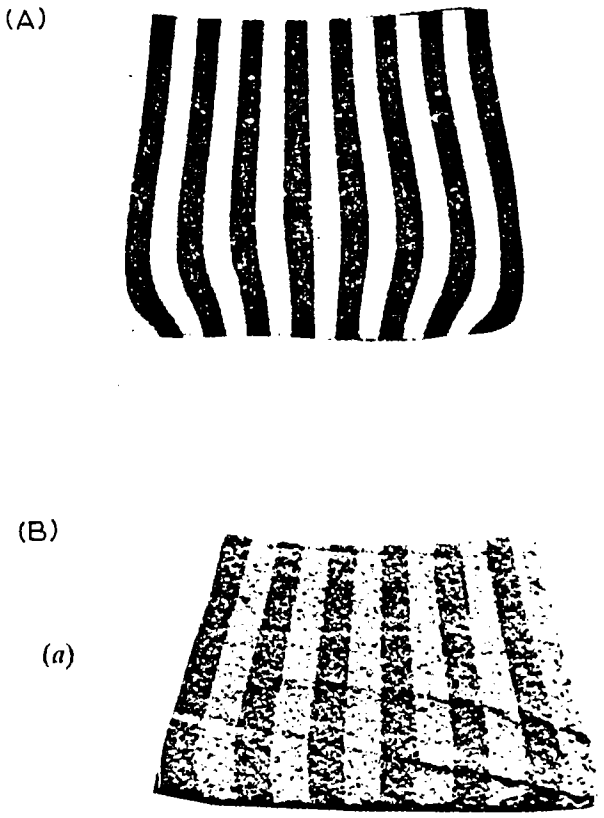

Fig. 6. A. Spreading under own weight of rectangular parallelpiped of viscous silicone putty $R = 0.5$. **Coherence at base. Early stage of deformation shown. Note similarity between early stage (A) and** stream-function simulation, Fig. 5. B. Spreading under own weight of rectangular parallelpiped of viscous silicone putty. $R = 0.5$. Free slip at base (silicone body floating on mercury). Note similarity with Fig. 4D.

Numerical results of the simulation

As numerical examples of the simulation of spreading double layer nappes the following series of models, consisting of a 50 m thick basal sheet (layer 1) overlain by a 5000 m thick layer 2 are treated. The density of both layers is kept at 2.8 g/cm^3 and the viscosity of layer 2 is kept at 10^{22} poise (10^{21} Pa s), while the viscosity of layer 1 varies from model to model. The instantaneous velocity field of models with

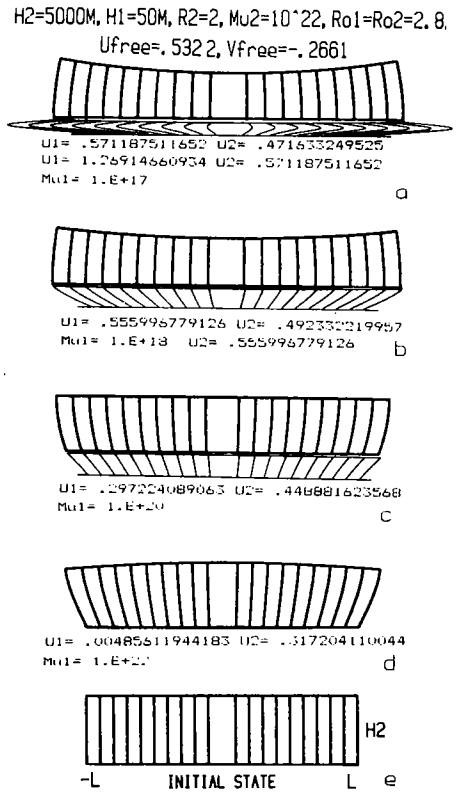

Fig. 7. Deformation after $5 \cdot 10^5$ years of creep of double-layer model with $R_2 = 2$ assuming constant initial velocity. Height of layer 1 in *a*, *b* and *c* is 50-fold exaggerated to show deformation of initially straight vertical markers. For explanation of symbols see Figs. 4 and 5; in addition: 10^22≡10^{22}, *Ufree* and *Vfree* are velocity components at the corners $y = H_2$, $x = \pm L$ if there was free slip at the base. When two values for u_1 and/or u_2 are given, one refers so the velocity at the layer contact at the front face, the other u_2 refers to the edge at $X = L$, $Y = H_2$, while the other u_1 refers to the middle level of layer 1. Note the change of deformed shape as a function of the viscosity of layer 1. With high viscosity of the basal layer u_2 has maximum value at the top of layer 2, and v_2 varies so as to generate a convex top boundary. Low *Mu*1 gives maximal u_2 at the base of layer 2 and maximal u_1 in the middle of layer 1. Because of horizontal tensile strain in layer 2 the top surface assumes a concave shape. For discussion of these effects of an especially mobile basal layer see text pp. 38, 39. All velocities in cm/yr.

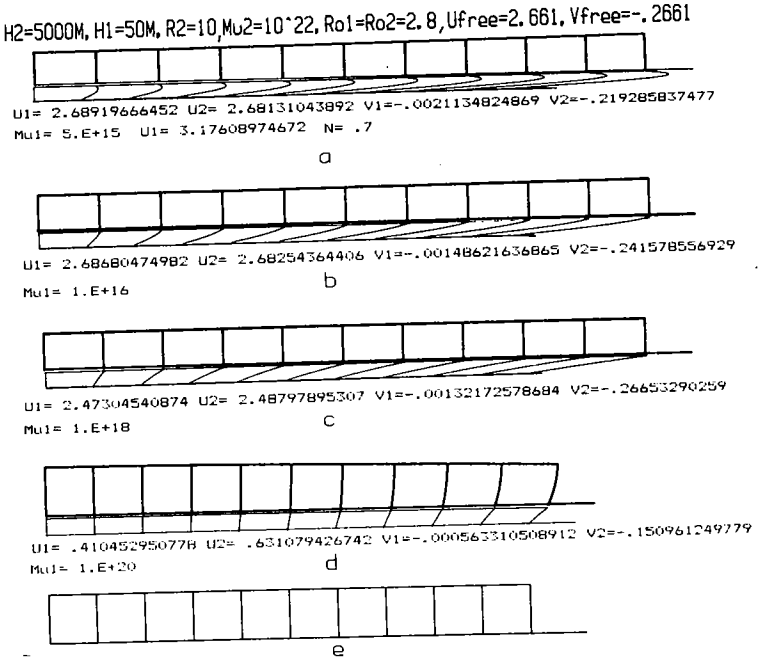

Fig. 8. Deformation after $5 \cdot 10^5$ years of creep assuming unchanged initial velocity of double lay aspect ratio $R_2 = 10$. Only right-hand half of model shown. Height of layer 1 exaggerated ten fol *b*, *c* and *d*. *e* is initial state. For explanation of symbols and definition of units, see Figs. 4 and that the strained profiles of the model show similarities but also significant differences when co with the shorter double layer in Fig. 7. The similarity is manifested by the increase of u_2 with h the front face (*d*), by the convex (though weak) top surface at low viscosity *Mu*l (d) and by excess of horizontal tensile strain in the back (middle) part of layer 2 (a). The difference is shown lack of variation of u_2 with height at the front face of layer 2 when *Mu*l is small, and by the co that in this model a lower viscosity *Mu*l is needed to produce "extrusion flow" in layer 1, i.e., velocity here being maximum at intermediate levels.

selected aspect ratios ($R_i = L/H_i$) and selected viscosity ratios ($Rm = \eta_1/\eta$ computed. The deformed shape of the initially rectangular cross section of mo shown, assuming that the initial instantaneous velocity was constant in the co an initial period of 500,000 years.

All pertinent results are shown in Figs. 4–15 and explained in the figure te instantaneous velocity at the front end of the models perhaps being th interesting result.

Extension of results by using simple scale-model rules

It is a consequence of scaling theory that the velocity field for mode thicknesses, viscosities and densities different from those computed may

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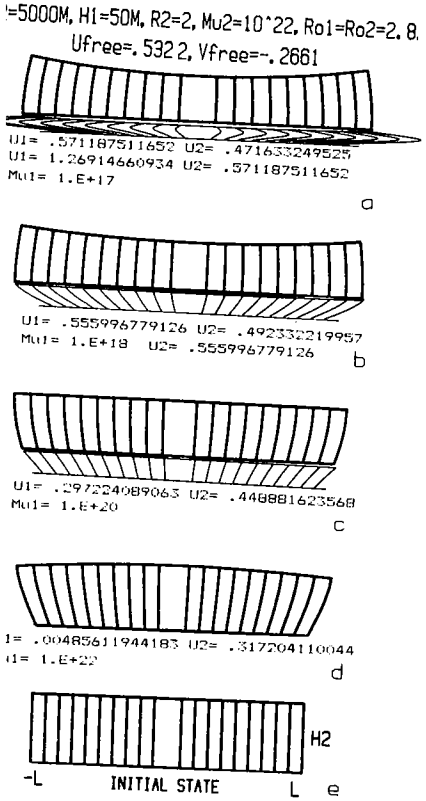


Deformation after $5 \cdot 10^5$ years of creep of double-layer model with $R_2 = 2$ assuming constant velocity. Height of layer 1 in *a*, *b* and *c* is 50-fold exaggerated to show deformation of initially vertical markers. For explanation of symbols see Figs. 4 and 5; in addition: 10^22≡10^{22}, *Ufree* ree are velocity components at the corners $y = H_2$, $x = \pm L$ if there was free slip at the base. wo values for u_1 and/or u_2 are given, one refers so the velocity at the layer contact at the front other u_2 refers to the edge at $X = L$, $Y = H_2$, while the other u_1 refers to the middle level of Note the change of deformed shape as a function of the viscosity of layer 1. With high viscosity asal layer u_2 has maximum value at the top of layer 2, and v_2 varies so as to generate a convex ndary. Low *Mu*1 gives maximal u_2 at the base of layer 2 and maximal u_1 in the middle of layer se of horizontal tensile strain in layer 2 the top surface assumes a concave shape. For discussion effects of an especially mobile basal layer see text pp. 38, 39. All velocities in cm/yr.

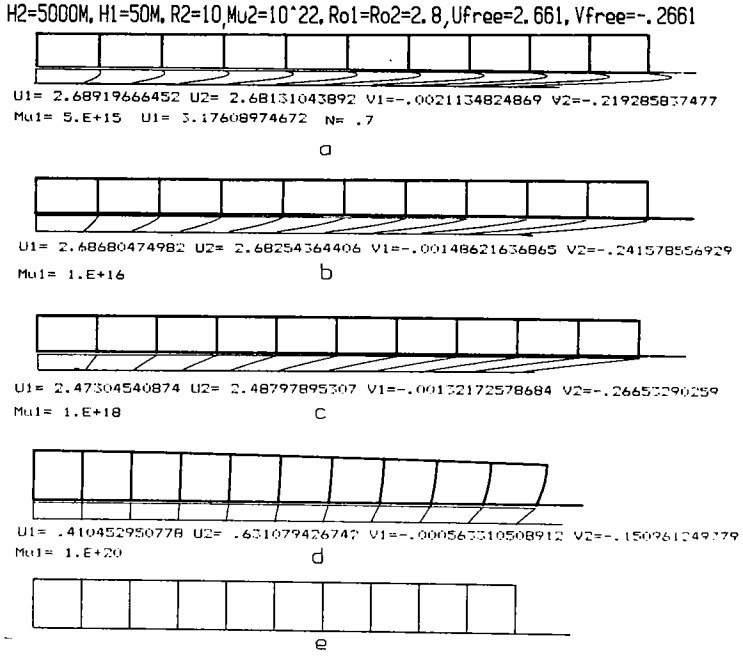


Fig. 8. Deformation after $5 \cdot 10^5$ years of creep assuming unchanged initial velocity of double layer with aspect ratio $R_2 = 10$. Only right-hand half of model shown. Height of layer 1 exaggerated ten fold in *a*, *b*, *c* and *d*. *e* is initial state. For explanation of symbols and definition of units, see Figs. 4 and 7. Note that the strained profiles of the model show similarities but also significant differences when compared with the shorter double layer in Fig. 7. The similarity is manifested by the increase of u_2 with height at the front face (*d*), by the convex (though weak) top surface at low viscosity *Mu*l (d) and by a weak excess of horizontal tensile strain in the back (middle) part of layer 2 (a). The difference is shown by the lack of variation of u_2 with height at the front face of layer 2 when *Mu*l is small, and by the condition that in this model a lower viscosity *Mu*l is needed to produce "extrusion flow" in layer 1, i.e., by the velocity here being maximum at intermediate levels.

selected aspect ratios ($R_i = L/H_i$) and selected viscosity ratios ($Rm = \eta_1/\eta_2$) are computed. The deformed shape of the initially rectangular cross section of models is shown, assuming that the initial instantaneous velocity was constant in the course of an initial period of 500,000 years.

All pertinent results are shown in Figs. 4–15 and explained in the figure texts, the instantaneous velocity at the front end of the models perhaps being the most interesting result.

Extension of results by using simple scale-model rules

It is a consequence of scaling theory that the velocity field for models with thicknesses, viscosities and densities different from those computed may be esti-

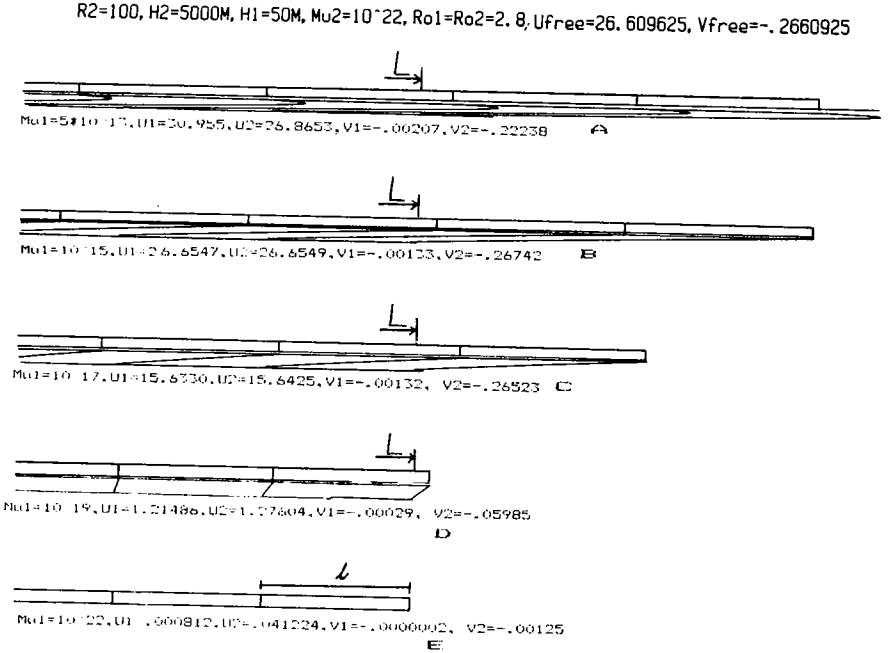


Fig. 9. Front part of spreading double-layer nappe with varying viscosity $Mu1$, showing initial velocity in cm/yr and displacement of front face after 500,000 yrs, assuming constant velocity. Thickness of layer 1 is 120 times exaggerated in order to show deformation of initially straight vertical markers. $l = 50$ km. L marks the initial position of the front face at 500 km from centre of the symmetrical nappe; viscosities $Mu1$ and $Mu2$ in poises.

mated by simple proportionality factors provided that the ratios between significant parameters are the same as for the models computed.

Thus, for example, when in a series of models all parameters are fixed except the absolute viscosities while, however, the viscosity ratio, R_m, is the same, then the velocity at any corresponding point is inversely proportional to the absolute viscosities.

Similarly, if all parameters in the models—including the density ratio—are fixed except the absolute densities, then the velocity at any corresponding points is proportional to the absolute densities.

For variations in geometric dimension the rule of thumb is: if in a series of models all parameters including aspect ratios and the ratio between thicknesses, are fixed except the absolute thicknesses and lengths, then the velocity at corresponding points varies with the square of a defined linear dimension,. e.g., the absolute thickness. In this case the strain rate at corresponding points varies linearly with the defined linear dimension (Table 2).

Using the above rules the information presented in Figs. 4–15 can be extended to

TABLE 2

Example demonstrating the second degree relation between geometric dimension and velocity. * R
$\rho_1 = \rho_2 = 2.8$ g/cm^3, $H_1/H_2 = 0.01$, $\eta_1 = 10^{18}$ P $\eta_2 = 10^{22}$ P

H_2 (m)	u_2 (cm/yr)
625	0.0864249627
1250	0.3456998510
2500	1.3827994038
5000	5.5311976153
10000	22.1247904614

* As may be readily checked, with a pocket calculator for example, these values show that a dou the linear dimension—in this case the height—corresponds to a four-fold increase of the veloci numerical relationship is very exact for the models under study.

any desired value of density, thickness and viscosity limited by the noted cons as regards parameter ratios.

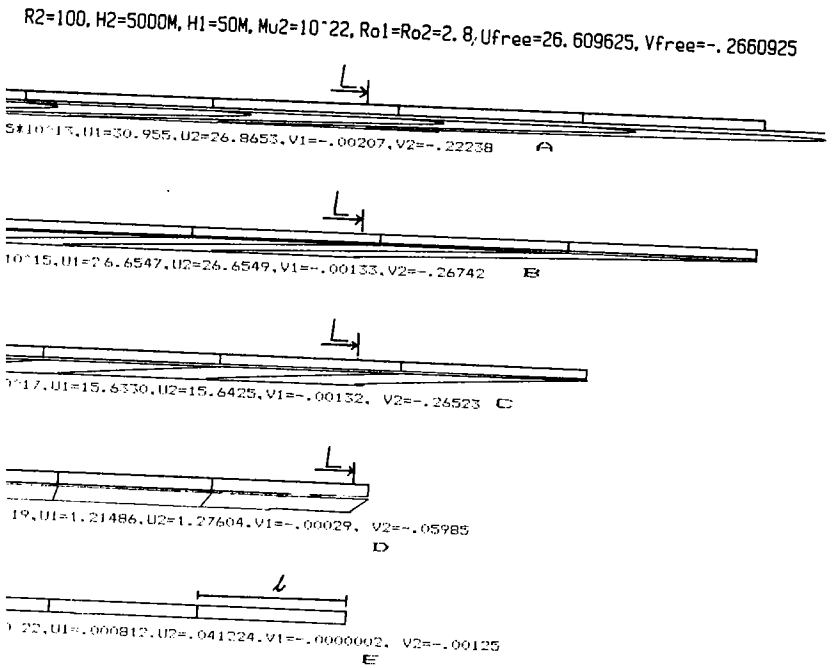
Strain in the basal layer

The rather complicated deformation of initially straight markers in laye demonstrated in most models studied requires further comments. We may that the strain in layer 1 is generated by the overlying layer since layer 1 is ge too thin to spread if not overlain by a heavy burden. The effect of layer 2 is tw (1) On account of its weight it squeezes the subjacent stratum and ge extrusion flow in the latter. This may be referred to as the "squeezing effect"; account of the gravitational spreading of layer 2, horizontal tensile strain transmitted to the underlying layer 1 since the contact is welded (coherent may appropriately be termed the "transmitted tension effect."

It is useful to be aware of these two distinct means by which layer 2 pr strain in layer 1. The two effects generally occur together, but that does not that they cannot be analysed separately. In fact, one of the effects may practice occur without any noticeable contribution from the other. If the su cumbent sheet is extremely viscous or rigid relative to the other it is unable t and spread, but its weight will still squeeze layer 1 and generate extrusio there.

It is more difficult to "purify" the "transmitted tension effect" in the t double-layer models here studied. The reason for this is that the relative viscosity required for gravitational spreading of layer 2 does of course not nu weight and its "squeezing effect". However, by selecting the proper viscosit between the two strata it is possible to construct numerical models in wh "transmitted tension" effect is the dominant one, see below.

In attempting to analyse the complex strain in layer 1 it helps to consi aforementioned two effects separately. When studying the "transmitted te



Front part of spreading double-layer nappe with varying viscosity *Mu*1, showing initial velocity in and displacement of front face after 500,000 yrs, assuming constant velocity. Thickness of layer 1 imes exaggerated in order to show deformation of initially straight vertical markers. $l = 50$ km. L he initial position of the front face at 500 km from centre of the symmetrical nappe; viscosities d *Mu*2 in poises.

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H_2 (m)	u_2 (cm/yr)
625	0.0864249627
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any desired value of density, thickness and viscosity limited by the noted constraints as regards parameter ratios.

Strain in the basal layer

The rather complicated deformation of initially straight markers in layer 1 as demonstrated in most models studied requires further comments. We may argue that the strain in layer 1 is generated by the overlying layer since layer 1 is generally too thin to spread if not overlain by a heavy burden. The effect of layer 2 is twofold: (1) On account of its weight it squeezes the subjacent stratum and generates extrusion flow in the latter. This may be referred to as the "squeezing effect"; (2) on account of the gravitational spreading of layer 2, horizontal tensile strain will be transmitted to the underlying layer 1 since the contact is welded (coherent). This may appropriately be termed the "transmitted tension effect."

It is useful to be aware of these two distinct means by which layer 2 produces strain in layer 1. The two effects generally occur together, but that does not mean that they cannot be analysed separately. In fact, one of the effects may well in practice occur without any noticeable contribution from the other. If the superincumbent sheet is extremely viscous or rigid relative to the other it is unable to yield and spread, but its weight will still squeeze layer 1 and generate extrusion flow there.

It is more difficult to "purify" the "transmitted tension effect" in the type of double-layer models here studied. The reason for this is that the relatively low viscosity required for gravitational spreading of layer 2 does of course not nullify its weight and its "squeezing effect". However, by selecting the proper viscosity ratio between the two strata it is possible to construct numerical models in which the "transmitted tension" effect is the dominant one, see below.

In attempting to analyse the complex strain in layer 1 it helps to consider the aforementioned two effects separately. When studying the "transmitted tension"

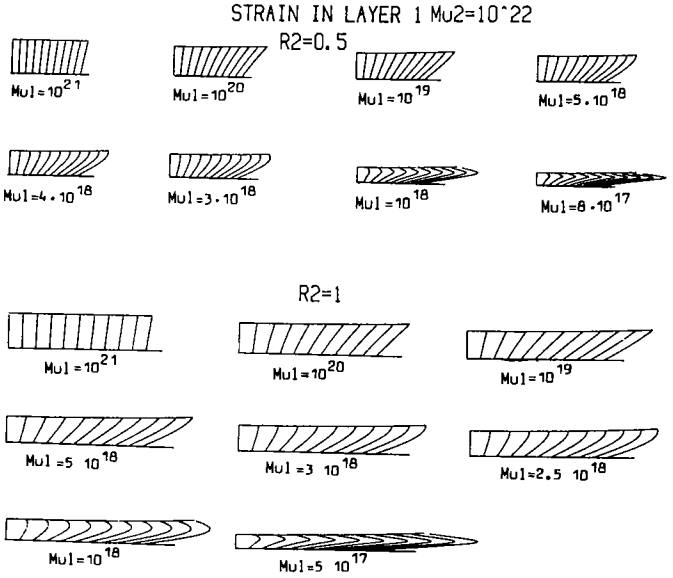
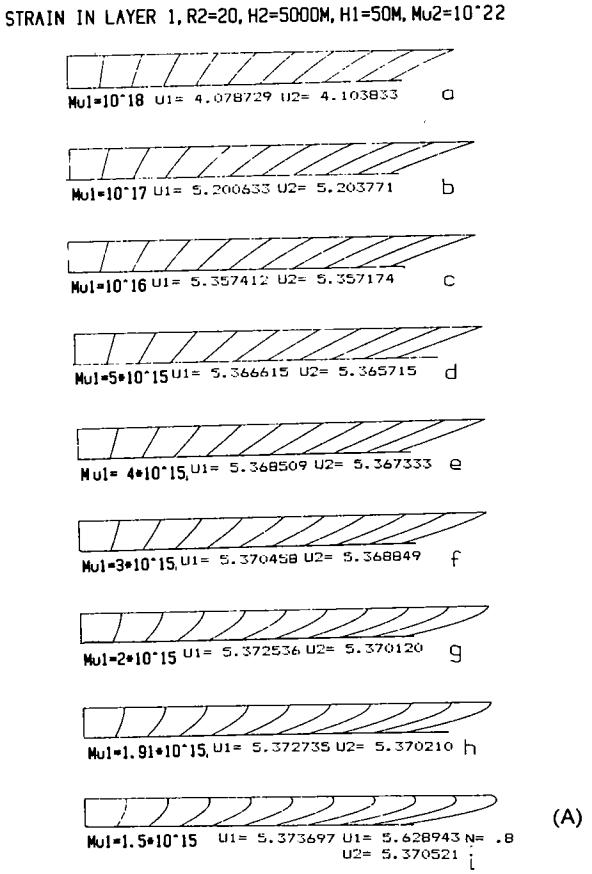


Fig. 10. Strain in layer 1 indicated by deformation of initially straight vertical markers shown at $R_2 = 0.5$ and 1, and for selected viscosity values in poises (P) of layer 1 while $Mu2$ is fixed at 10^{22} P. Note change of deformation with varying viscosity. When $Mu1$ is large the tilted markers are straight or show a gentle curvature, convex toward the (upper) left. With decreasing viscosity the curvature increases but now convex toward the (lower) right. As $Mu1$ decreases further the curvature sharpens and becomes pointed at the same time as the deformed markers below and above the sharp kinks bend gently in the opposite direction to the kinks. Note that the viscosity at the stage for "kinking" of the markers is less for aspect ratio $R_2 = 1$ than for $R_2 = 0.5$. Compare also Figs. 11 and 12. See comment in text, p. 233 ff.

effect, we cancel out the squeezing effect by disregarding the weight of layer 2, but at the same time assuming a contact-parallel longitudinal strain identical to that which would have been produced by the gravitational spreading. Since the contact is coherent the longitudinal extension is the same on either side of the contact, and it follows that in layer 1 longitudinal layer-parallel stretch increases from zero at the



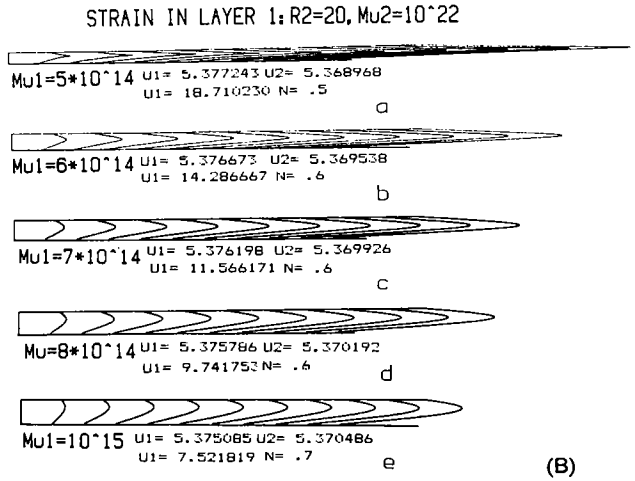


Fig. 11. A. Strain in layer 1 for selected viscosities at aspect ratio $R_2 = 20$ which corresponds to $R_1 = 2000$ since $H_1 = H_2/100$ and the layers are equally long. $Mu2$ is constant $= 10^{22}$ P. The switch of marker curvature (see Fig. 10) occurs at $Mu1 \approx 5 \cdot 10^5$ P compared with between 10^{19} and 10^{20} poise for $R_2 = 0.5$ and about 10^{19} for $R_2 = 1$. Velocities given in cm/yr at front end, $u2$ refers to upper right corner of layer 2 (not shown in figure), and $u1$ to the upper right corner of layer 1, i.e., to the contact between the layers, and for $Mu1 = 1.5 \cdot 10^5$ also at level $N_1 = 0.8$ where $u1$ reaches maximal value. B. Same as 11A but the effect of additional viscosity values is demonstrated. Note especially the extremely sharp cusps in the deformed markers at $Mu1 = 5 \cdot 10^{14}$ P. It is also worth noting that the level N at which maximum $u1$ is reached moves close to the centre when viscosity decreases. N gives the level in tenths of $H1$.

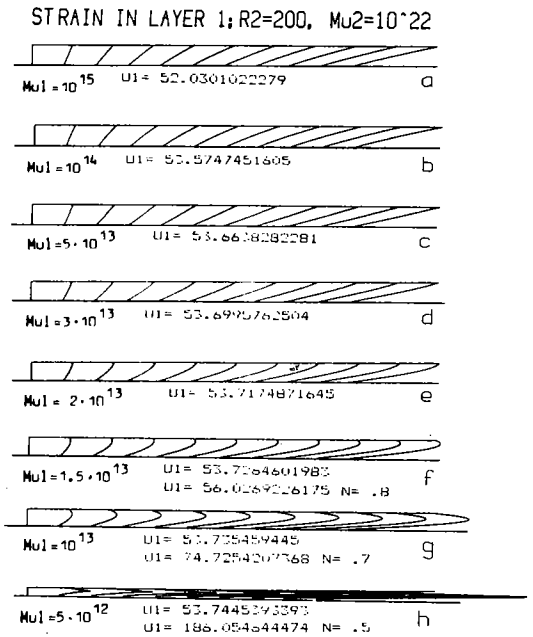


Fig. 12. With the extreme aspect of layer 1 in this example ($R_1 = 20{,}000$) the switch of curvature in the deformed markers occurs at Mul between $3 \cdot 10^{13}$ and $5 \cdot 10^{13}$ P, and the sharp cusps develop at $Mul = 5 \cdot 10^{12}$P; compare Figs. 9 and 10. ul is given in cm/year at front face. For $Mul \leqq 1.5 \cdot 10^{13}$ maximum ul is also given, occurring at levels $N = 0.8$, 0.7 and 0.5, respectively. It is interesting that the computer-plotted figure indicates that shear at layer contact vanishes (the markers are normal to the top surface of layer 1) when Mul is between $1.5 \cdot 10^{13}$ and $2 \cdot 10^{13}$ P (see also Table 3).

contact with the rigid base to maximal value at the contact to the spreading layer 2. It can be demonstrated that a gradient across the layer of layer-parallel longitudinal strain causes initially straight passive markers not only to tilt but also to bend with the concave side of the curvature facing in the direction of motion, (see Fig. 13). This effect is detectable in some of the models computed in which the viscosity ratio happened to be sufficiently favourable for a certain dominance of "the transmitted tension" effect. This would mean relatively high viscosity of layer 1, thereby reducing the squeezing effect (see, e.g., Figs. 8, 10 and 11). However, the models studied were intended to illustrate the "lubrication" tendency of a relatively soft basal sheet in the problem of thrust motion. Hence all models computed so far have basal layers less viscous than the superincumbent layer. For illustration of the "transmitted tension" phenomenon it is desirable to select viscosities for the basal stratum which are higher than the viscosity of the layer above. An example of this is presented in Fig. 15. In this figure not only does the longitudinal strain increase across the stratum from base to top, but there is also a positive gradient of the longitudinal strain along the layer from back to front. This means that the gradient

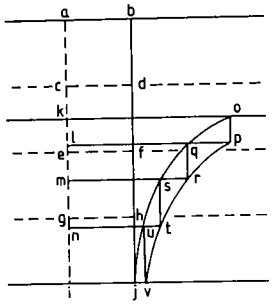


Fig. 13. Deformation of a straight marker (b to j) caused by a gradient parallel to the marke longitudinal tensile strain normal to the marker. For demonstration the strain increases stepwise a the gradient, being small in the region *ghij*, larger in the region *efgh*, still larger in the region *cdef* assuming maximum value in the region *abcd*. The regions mentioned belong to zones extending no to the marker, the strain within each zone being homogeneous and the thickness of the z infinitesimal. The line $a-i$ is in the centre of the strain so there is no motion across this line. In stepwise increasing strain the region *ghij* changes to *nuiv*, the region *efgh* to *msnt*, the region *cd eqmr*, and ultimately the region *abcd* goes to *kolp*. It is clear that if this operation was made infinitesimal steps the marker *bj* will deform to a continuous smooth curve within the region bounde *prtv* and *qsuj*.

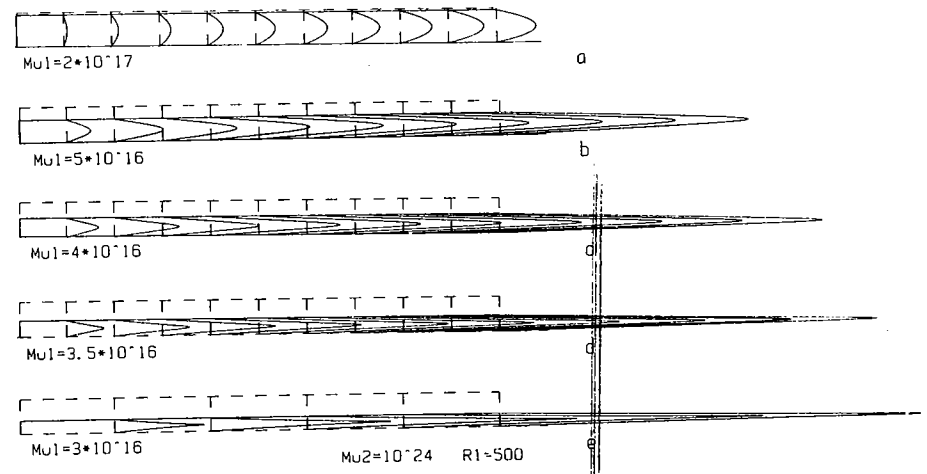
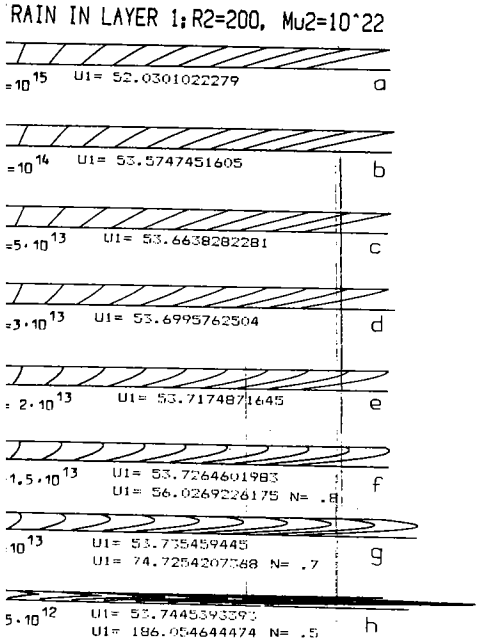


Fig. 14. Extrusion flow in layer 1 caused by the "squeezing effect" of layer 2 (not shown in the fi Layer 2 is given the viscosity $\eta_2 = 10^{24}$ P to make it practically immobile. Note the well deve symmetry of the flow in contrast to models where the "squeezing effect" is mixed with the "transi tension effect" (Figs. 10, 11 and 12). To show the well developed sharp cusps in the markers at squeezing only every second marker is recorded. Dashed lines give initial outline of cross sectio indicate the initial markers. Deformed pattern based on initial velocity remaining unchanged for yrs.



2. With the extreme aspect of layer 1 in this example ($R_1 = 20{,}000$) the switch of curvature in the ned markers occurs at $Mu1$ between $3\cdot10^{13}$ and $5\cdot10^{13}$ P, and the sharp cusps develop at $5\cdot10^{12}$P; compare Figs. 9 and 10. $u1$ is given in cm/year at front face. For $Mu1 \leqq 1.5\cdot10^{13}$ ium $u1$ is also given, occurring at levels $N = 0.8$, 0.7 and 0.5, respectively. It is interesting that the ter-plotted figure indicates that shear at layer contact vanishes (the markers are normal to the top : of layer 1) when $Mu1$ is between $1.5\cdot10^{13}$ and $2\cdot10^{13}$ P (see also Table 3).

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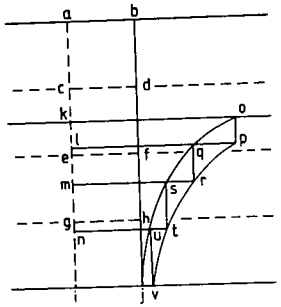
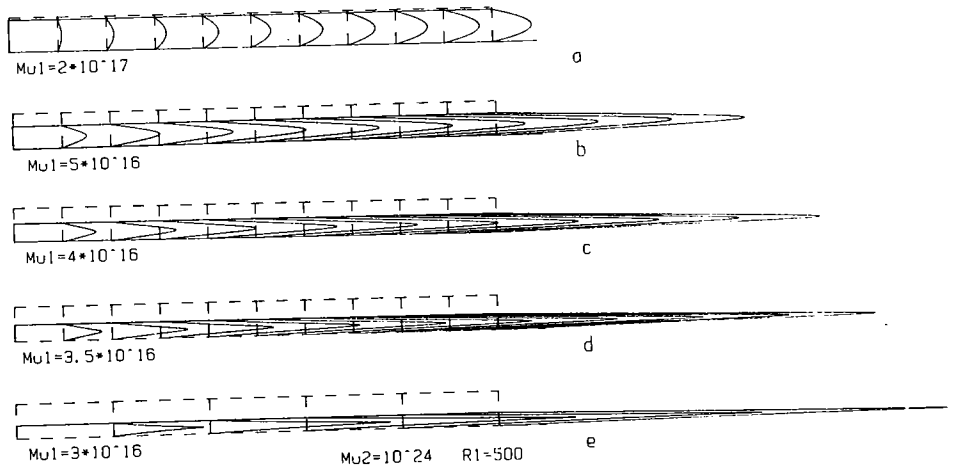


Fig. 13. Deformation of a straight marker (b to j) caused by a gradient parallel to the marker of longitudinal tensile strain normal to the marker. For demonstration the strain increases stepwise along the gradient, being small in the region *ghij*, larger in the region *efgh*, still larger in the region *cdef* and assuming maximum value in the region *abcd*. The regions mentioned belong to zones extending normal to the marker, the strain within each zone being homogeneous and the thickness of the zones infinitesimal. The line $a-i$ is in the centre of the strain so there is no motion across this line. In this stepwise increasing strain the region *ghij* changes to *nuiv*, the region *efgh* to *mstn*, the region *cdef* to *eqmr*, and ultimately the region *abcd* goes to *kolp*. It is clear that if this operation was made with infinitesimal steps the marker *hj* will deform to a continuous smooth curve within the region bounded by *prtv* and *qsuj*.



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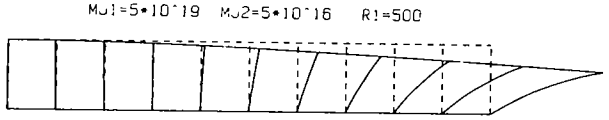

Fig. 15. The "transmitted tension effect" in layer 1 caused by horizontal spreading of layer 2 (not shown) and the transmission of longitudinal strain across the coherent boundary. Note the curvature in the deformed markers as caused by the strain increasing from bottom to top as well as from the back to the front. The vertical dimension is exaggerated by a factor of 90 in this figure. Compare with Fig. 13.

of the strain across the layer becomes steeper as the front is approached (since strain is zero at the base); a condition which explains why the curvature of the deformed markers is accentuated in the frontal region. (Incidentally, the increment of layer-parallel stretch in the frontal region in layer 1 is in good agreement with the strain in layer 2 in models with relatively competent basal layers, see, e.g., Figs. 11 and 12.) As pointed out above, the "squeezing effect" can be readily demonstrated in its "pure" state by selecting a rigid or highly viscous overburden. With vanishing spreading of the overburden layer-parallel stretch is zero both at the top and bottom of layer 1 and maximum tensile strain will be in the middle of the layer. This is well demonstrated in Fig. 14 in which the perfect symmetry of the "bulge" of deformed markers should be compared with the unsymmetric "bulge" in models allowing squeezing and tension transmission to occur with comparable intensity.

It is clear from the above discussion of the longitudinal strain gradient that the curvature and the extreme cusps on the markers in Fig. 14 and other examples arise from the gradients from both edges to the centre of layer parallel longitudinal strain.

At certain relations between the viscosity ratio and the aspect ratio of the model the shear strain vanishes at the boundary between the two layers; see Table 3. When this happens the second layer moves as if there was no friction at the base. If the

TABLE 3

Relation between aspect ratio R_2 and viscosity ratio η_1/η_2 at which the shear stress vanishes at layer 1/layer 2 contact

R_2	η_1/η_2	η_1	η_2
0.5	$4.0\cdot10^{-4}$	$4\cdot10^{18}$	10^{22}
1	$2.5\cdot10^{-4}$	$2.5\cdot10^{18}$	10^{22}
2	$4.3\cdot10^{-5}$	$4.3\cdot10^{17}$	10^{22}
5	$3.5\cdot10^{-6}$	$3.5\cdot10^{16}$	10^{22}
10	$7.6\cdot10^{-7}$	$7.6\cdot10^{15}$	10^{22}
20	$1.85\cdot10^{-7}$	$1.85\cdot10^{15}$	10^{22}
50	$2.9\cdot10^{-8}$	$2.9\cdot10^{14}$	10^{22}
100	$7.4\cdot10^{-9}$	$7.4\cdot10^{13}$	10^{22}
200	$1.9\cdot10^{-9}$	$1.9\cdot10^{13}$	10^{22}

viscosity of layer 1 is less than that given by this critical value, the flow in lay generates a horizontal pull on layer 2, the pull being maximal in the centre of nappe.

PROSPECTED DEVELOPMENT OF A STREAM FUNCTION-ENERGY EXTREMIZ METHOD

It is obviously a serious limitation of the combined "stream function-energy tremizing" method presented here that only the instantaneous velocity fiel calculated. The goal for a user of the method must be to follow the com evolution of the deforming process—in our case the complete history of spreading viscous "nappe". It is easy to see that this may be accomplished step-wise procedure, assuming that the velocity field is constant for a limited interval and then calculating a new velocity field based on the initially defo model, the new velocity field giving rise to further deformation which in its produces an altered velocity field that is used to calculate the next ste deformation, etc., etc. The new velocity field characterizing a deformed state o model is based on new values of the coefficients in the polynomial stream func To obtain the new values of the coefficients strain- and potential energy mu calculated by integration over the deformed cross section of the model, usin formula for the "specific energy rate" as regards the strain energy, but calcul the potential energy change by a method different from the one used above fo infinitesimal instantaneous state. Now the potential energy must be calculated a *end* of the first step, using the formula:

$$E_{\text{pot1}} = \rho g \int\int y \partial x \partial y$$

the boundary of the deformation profile being defined by the collection of poi y given by the formulas:

$$x = x_i + u\Delta t$$

and:

$$y = y_i + v\Delta t$$

Here x_i and y_i are position vectors for the initial profile boundary and the ve components u and v are derived from the stream function whose coefficients not yet been determined. Δt is an arbitrary short time interval.

The change of potential energy during the interval Δt is then:

$$\Delta E_{\text{pot}} = E_{\text{pot1}} - E_{\text{pot0}}$$

where:

$$E_{\text{pot0}} = \rho g \int\int y \partial x \partial y$$

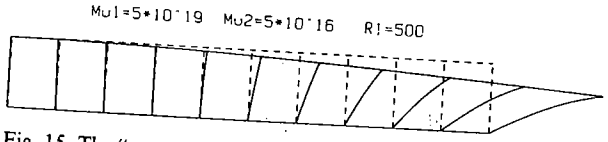

Fig. 15. The "transmitted tension effect" in layer 1 caused by horizontal spreading of layer 2 (not shown) and the transmission of longitudinal strain across the coherent boundary. Note the curvature in the deformed markers as caused by the strain increasing from bottom to top as well as from the back to the front. The vertical dimension is exaggerated by a factor of 90 in this figure. Compare with Fig. 13.

of the strain across the layer becomes steeper as the front is approached (since strain is zero at the base); a condition which explains why the curvature of the deformed markers is accentuated in the frontal region. (Incidentally, the increment of layer-parallel stretch in the frontal region in layer 1 is in good agreement with the strain in layer 2 in models with relatively competent basal layers, see, e.g., Figs. 11 and 12.) As pointed out above, the "squeezing effect" can be readily demonstrated in its "pure" state by selecting a rigid or highly viscous overburden. With vanishing spreading of the overburden layer-parallel stretch is zero both at the top and bottom of layer 1 and maximum tensile strain will be in the middle of the layer. This is well demonstrated in Fig. 14 in which the perfect symmetry of the "bulge" of deformed markers should be compared with the unsymmetric "bulge" in models allowing squeezing and tension transmission to occur with comparable intensity.

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ABLE 3

lation between aspect ratio R_2 and viscosity ratio η_1/η_2 at which the shear stress vanishes at layer layer 2 contact

	η_1/η_2	η_1	η_2
.5	$4.0 \cdot 10^{-4}$	$4 \cdot 10^{18}$	10^{22}
	$2.5 \cdot 10^{-4}$	$2.5 \cdot 10^{18}$	10^{22}
	$4.3 \cdot 10^{-5}$	$4.3 \cdot 10^{17}$	10^{22}
	$3.5 \cdot 10^{-6}$	$3.5 \cdot 10^{16}$	10^{22}
	$7.6 \cdot 10^{-7}$	$7.6 \cdot 10^{15}$	10^{22}
	$1.85 \cdot 10^{-7}$	$1.85 \cdot 10^{15}$	10^{22}
	$2.9 \cdot 10^{-8}$	$2.9 \cdot 10^{14}$	10^{22}
	$7.4 \cdot 10^{-9}$	$7.4 \cdot 10^{13}$	10^{22}
	$1.9 \cdot 10^{-9}$	$1.9 \cdot 10^{13}$	10^{22}

viscosity of layer 1 is less than that given by this critical value, the flow in layer 1 generates a horizontal pull on layer 2, the pull being maximal in the centre of the nappe.

PROSPECTED DEVELOPMENT OF A STREAM FUNCTION-ENERGY EXTREMIZING METHOD

It is obviously a serious limitation of the combined "stream function-energy-extremizing" method presented here that only the instantaneous velocity field is calculated. The goal for a user of the method must be to follow the complete evolution of the deforming process—in our case the complete history of the spreading viscous "nappe". It is easy to see that this may be accomplished by a step-wise procedure, assuming that the velocity field is constant for a limited time interval and then calculating a new velocity field based on the initially deformed model, the new velocity field giving rise to further deformation which in its turn produces an altered velocity field that is used to calculate the next step of deformation, etc., etc. The new velocity field characterizing a deformed state of the model is based on new values of the coefficients in the polynomial stream function. To obtain the new values of the coefficients strain- and potential energy must be calculated by integration over the deformed cross section of the model, using the formula for the "specific energy rate" as regards the strain energy, but calculating the potential energy change by a method different from the one used above for the infinitesimal instantaneous state. Now the potential energy must be calculated at the *end* of the first step, using the formula:

$$E_{\text{pot1}} = \rho g \int\int y \partial x \partial y \tag{95}$$

the boundary of the deformation profile being defined by the collection of points x, y given by the formulas:

$$x = x_i + u\Delta t \tag{96}$$

and:

$$y = y_i + v\Delta t \tag{97}$$

Here x_i and y_i are position vectors for the initial profile boundary and the velocity components u and v are derived from the stream function whose coefficients have not yet been determined. Δt is an arbitrary short time interval.

The change of potential energy during the interval Δt is then:

$$\Delta E_{\text{pot}} = E_{\text{pot1}} - E_{\text{pot0}} \tag{98}$$

where:

$$E_{\text{pot0}} = \rho g \int\int y \partial x \partial y \tag{99}$$

means integration over the initial profile. ΔE_{pot} is to be equated with $\dot{E}_{strain}\Delta t$, and the optimizing method described above (keeping Δt constant) leads to determination of the coefficients in the stream function.

Tests have shown that this procedure leads to complications when it comes to optimizing the rate of energy change in attempts to determine the new coefficients in the stream function. Work is, however, in progress trying to solve this problem.

APPENDIX A

Applying the expressions:

$$\epsilon = \frac{\partial u}{\partial x} \quad \text{and} \quad \gamma = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

to eqn. (34) we obtain the strain energy rate per unit volume:

$$\dot{e}_{\epsilon\gamma_1} = \eta_1\left(4\epsilon_x^2 + \gamma_{xy}^2\right) \tag{A1}$$

which upon integration over the cross section $0 \leqq x \leqq L$, $0 \leqq y \leqq H_1$ gives the strain energy rate for that part of the cross section slice which cuts layer 1:

$$\begin{aligned}\dot{E}_{\epsilon\gamma_1} = \eta_1\Big[&\left(\tfrac{16}{3}R_1 + \tfrac{4}{3}R_1^3\right)H_1^4a^2 + \left(\tfrac{36}{5}R_1 + \tfrac{12}{3}R_1^3\right)H_1^6b^2 \\ &+ \left(\tfrac{64}{7}R_1 + \tfrac{132}{15}R_1^3 + \tfrac{72}{15}R_1^5 + \tfrac{4}{7}R_1^7\right)H^8c^2 + \left(\tfrac{100}{9}R_1^3 + \tfrac{100}{7}R_1^3 + 12R_1^5\right. \\ &\left.+ \tfrac{100}{21}R_1^7\right)H^{10}d^2 + \left(12R_1 + 4R_1^3\right)H_1^5ab + \left(\tfrac{64}{5}R_1 - \tfrac{24}{9}R_1^3 - \tfrac{8}{5}R_1^5\right)H_1^6ac \\ &+ \left(\tfrac{40}{3}R_1 - 10R_1^3 - 4R_1^5\right)H_1^7ad + \left(16R_1 + 6R_1^3 - \tfrac{24}{10}R_1^5\right)H_1^7bc \\ &+ \left(\tfrac{120}{7}R_1 - 8R_1^5\right)H_1^8bd + \left(20R_1 + 20R_1^3 + 15R_1^5 + \tfrac{20}{7}R_1^7\right)H_1^9cd\Big]\end{aligned} \tag{A2}$$

In the same way the energy of the part of the cross section slice which cuts through layer 2 is derived from stream function (35):

$$\begin{aligned}\dot{E}_{\epsilon\gamma_2} = \eta_2\Big[&4a_{22}^2H_2^2R_2 + \left(\tfrac{16}{3}R_2 + \tfrac{4}{3}R_2^3\right)a_{23}^2H_2^4 + \left(\tfrac{36}{5}R_2 + 4R_2^3\right)a_{24}^2H_2^6 \\ &+ \left(\tfrac{64}{7}R_2 + \tfrac{44}{5}R_2^3 + \tfrac{24}{5}R_2^5 + \tfrac{4}{7}R_2^7\right)a_{25}^2H_2^8 + \left(\tfrac{100}{9}R_2 + \tfrac{100}{7}R_2^3\right. \\ &\left.+ 12R_2^5 + \tfrac{100}{21}R_2^7\right)a_{26}^2H_2^{10} + 12R_2^3a_{41}^2H_2^4 + \left(4R_2^3 + \tfrac{36}{5}R_2^5\right)a_{42}^2H_2^6 \\ &+ a_{22}\left(8a_{23}R_2H_2^3 + 8a_{24}R_2H_2^4 + 8\left(R_2 - R_2^3\right)a_{25}H_2^5 + \left(8R_2 - \tfrac{40}{3}R_2^3\right)a_{26}H_2^6\right. \\ &\left.+ 8R_2^3a_{42}H_2^4\right) + a_{23}\left[\left(12R_2 + 4R_2^3\right)a_{24}H_2^5 + \left(\tfrac{64}{5}R_2 - \tfrac{8}{3}R_2^3 - \tfrac{8}{5}R_2^5\right)a_{25}H_2^6\right. \\ &\left.+ \left(\tfrac{40}{3}R_2 - 10R_2^3 - 4R_2^5\right)a_{26}H_2^7 - 8R^3a_{41}H_2^4 + 4R_2^3a_{42}H_2^5\right] \\ &+ a_{24}\left\{\left(16R_2 + 6R_2^3 - \tfrac{12}{5}R_2^5\right)a_{25}H_2^7 + \left(\tfrac{120}{7}R_2 - 8R_2^5\right)a_{26}H_2^8\right. \\ &\left.- 12R_2^3a_{41}H_2^5\right\} + a_{25}\left\{\left(20R_2 + 20R_2^3 + 12R_2^5 + \tfrac{20}{7}R_2^7\right)a_{26}H_2^9\right. \\ &\left.- \left(10R_2^3 + 12R_2^5\right)a_{42}H_2^7 - \left(24R_2^3 - \tfrac{24}{5}R_2^5\right)a_{41}H_2^6\right\} \\ &+ a_{26}\left\{-\left(16R_2^3 + 16R_2^5\right)a_{42}H_2^8 - \left(30R_2^3 - 12R_2^5\right)a_{41}H_2^7\right\} \\ &+ 12R_2^3a_{41}a_{42}H_2^5\Big]\end{aligned} \tag{A3}$$

The strain energy rate for the total cross section slice of unit thickness thro the model is then:

$$\dot{E}_{\epsilon\gamma} = \dot{E}_{\epsilon\gamma_1} + \dot{E}_{\epsilon\gamma_2}$$

The rate of change of potential energy for the unit slice follows upon integra of the expression:

$$\dot{e}_{pot} = \rho_i g v_i \partial x \partial y \qquad \text{(see also p. 239).}$$

across layer 1 when i is 1 and layer 2 when $i = 2$. Here the vertical velc component is derived from the appropriate stream functions (34) or (35). procedure leads to the potential energy rate for the whole unit section across model:

$$\begin{aligned}\dot{E}_{pot} = \dot{E}_{pot1} + \dot{E}_{pot2} = \rho_1 g\Big[&-\left(\tfrac{1}{3}H_1^4a + \tfrac{1}{4}H_1^5b + \tfrac{1}{5}H_1^6c + \tfrac{1}{6}H_1^7d\right)R_1 \\ &+ \left(\tfrac{1}{3}H_1^6c + \tfrac{5}{12}H_1^7d\right)R_1^3\Big] + \rho_2 g\Big[\left(H_2^2a_{21} + \tfrac{1}{2}H_2^3a_{22} + \tfrac{1}{3}H_2^4a_{23}\right. \\ &\left.+ \tfrac{1}{4}H_2^5a_{24} + \tfrac{1}{5}H_2^6a_{25} + \tfrac{1}{6}H_2^7a_{26}\right)R_2 + \left(H_2^4a_{41} + \tfrac{1}{2}H_2^5a_{42}\right. \\ &\left.- \tfrac{1}{3}H_2^6a_{25} - \tfrac{5}{12}H_2^7a_{26}\right)R_2^3\Big]\end{aligned}$$

R_i ($i = 1$ to 2) is the aspect ratio, L/H_i of the layers.

APPENDIX B

```
10   !"COMB86NAPW"
20 !A plotting,printing program for double-layer spreading nappe with vanishir
shear stress at top surface;valid for initial instantaneous velocities
30 !Theory and program (HPbasic,2,1;computer HP9816s) By Hans Ramberg,Uppsala
40 !Adjustment-if any-of viscosity,layer thickness,density (lines490,600) or p
 velocity (line 230) must be don prior to pressing "RUN"
50 !Scale values (line 280) are generally input after start but may even be ir
rted prior to start
60 !To start program put on the plotter and press "RUN"
70 !!
80 !The two polynomial STREAM FUNCTIONS are:
90 !Psi1=-(AY^2+BY^3+CY^4+DY^5+EY^6+FY^7)X+(Cy^2+5/3Dy^3+10/3Ey^4+14/3Fy^5)x^
Ey^2+7/3Fy^3)x^5
100 !Psi2=(A21+A22y+A23y^2+A24y^3+A25y^4+A26y^5)x+(A41+A42y-(A25+5*A61)y^2-5/
A26+A62)y^3)x^3+(A61+A62y)x^5
110 !!
120 !H1,H2 are thicknesses,Rh=H1/H2,L= length,R2(INPUT item!)=R=L/H2,R1=L/H1,
,Ro2 are densities,Rro=Ro1/Ro2,Mu1,Mu2 are viscosities,Rm(INPUT item!)=Mu1/Mu
130 !!
140! Gr=acceleration of gravity, U1,V1,U2,V2 are horizontal (U) and vertical
 velocity components in layer 1 and 2
150 !!
160 !Ufree,Vfree are velocitis at front face of free-slip model;X1,Y1;X2,Y2 a
position vectors in layer 1,resp.layer 2;Mi=X/L,Ni=Y/H in lines 1470,1490 ff.
170 ! are normalized position vectors
180 !!
190 ! If hight is in cm, density in g/cm^3, Gr in cm/s^2  and viscosity in po
then velocity is in cm/year;with hight in m, density in kg/m^3, viscosity in
200 !Pascal s and Gr in m/s^2 then velocity is in meter per year
210 !!
220  PRINTER IS 705 !(705 is really a plotter:HP7470A)
230  PRINT "IN;IP,VS2;"!This prepares for PLOTTING, and specifies PEN VELOCIT
240  TRACE PAUSE
```

ıeans integration over the initial profile. ΔE_{pot} is to be equated with $\dot{E}_{strain}\Delta t$, and ıe optimizing method described above (keeping Δt constant) leads to determina- ɔn of the coefficients in the stream function.

Tests have shown that this procedure leads to complications when it comes to ɔtimizing the rate of energy change in attempts to determine the new coefficients the stream function. Work is, however, in progress trying to solve this problem.

PENDIX A

Applying the expressions:

$$\frac{\partial u}{\partial x} \quad \text{and} \quad \gamma = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

ɛqn. (34) we obtain the strain energy rate per unit volume:

$$= \eta_1\left(4\epsilon_x^2 + \gamma_{xy}^2\right) \tag{A1}$$

ch upon integration over the cross section $0 \leqq x \leqq L$, $0 \leqq y \leqq H_1$ gives the strain gy rate for that part of the cross section slice which cuts layer 1:

$$\begin{aligned}
= \eta_1\Big[&\left(\tfrac{16}{3}R_1 + \tfrac{4}{3}R_1^3\right)H_1^4a^2 + \left(\tfrac{36}{5}R_1 + \tfrac{12}{3}R_1^3\right)H_1^6b^2 \\
&+ \left(\tfrac{64}{7}R_1 + \tfrac{132}{15}R_1^3 + \tfrac{72}{15}R_1^5 + \tfrac{4}{7}R_1^7\right)H^8c^2 + \left(\tfrac{100}{9}R_1^3 + \tfrac{100}{7}R_1^3 + 12R_1^5\right. \\
&\left. + \tfrac{100}{21}R_1^7\right)H^{10}d^2 + \left(12R_1 + 4R_1^3\right)H_1^5ab + \left(\tfrac{64}{5}R_1 - \tfrac{24}{9}R_1^3 - \tfrac{8}{5}R_1^5\right)H_1^6ac \\
&+ \left(\tfrac{40}{3}R_1 - 10R_1^3 - 4R_1^5\right)H_1^7ad + \left(16R_1 + 6R_1^3 - \tfrac{24}{10}R_1^5\right)H_1^7bc \\
&+ \left(\tfrac{120}{7}R_1 - 8R_1^5\right)H_1^8bd + \left(20R_1 + 20R_1^3 + 15R_1^5 + \tfrac{20}{7}R_1^7\right)H_1^9cd\Big]
\end{aligned} \tag{A2}$$

the same way the energy of the part of the cross section slice which cuts gh layer 2 is derived from stream function (35):

$$\begin{aligned}
= \eta_2\Big[&4a_{22}^2H_2^2R_2 + \left(\tfrac{16}{3}R_2 + \tfrac{4}{3}R_2^3\right)a_{23}^2H_2^4 + \left(\tfrac{36}{5}R_2 + 4R_2^3\right)a_{24}^2H_2^6 \\
&+ \left(\tfrac{64}{7}R_2 + \tfrac{44}{5}R_2^3 + \tfrac{24}{5}R_2^5 + \tfrac{4}{7}R_2^7\right)a_{25}^2H_2^8 + \left(\tfrac{100}{9}R_2 + \tfrac{100}{7}R_2^3\right. \\
&\left. + 12R_2^5 + \tfrac{100}{21}R_2^7\right)a_{26}^2H_2^{10} + 12R_2^3a_{41}^2H_2^4 + \left(4R_2^3 + \tfrac{36}{5}R_2^5\right)a_{42}^2H_2^6 \\
&+ a_{22}\left(8a_{23}R_2H_2^3 + 8a_{24}R_2H_2^4 + 8\left(R_2 - R_2^3\right)a_{25}H_2^5 + \left(8R_2 - \tfrac{40}{3}R_2^3\right)a_{26}H_2^6\right. \\
&\left. + 8R_2^3a_{42}H_2^4\right) + a_{23}\Big[\left(12R_2 + 4R_2^3\right)a_{24}H_2^5 + \left(\tfrac{64}{5}R_2 - \tfrac{8}{3}R_2^3 - \tfrac{8}{5}R_2^5\right)a_{25}H_2^6 \\
&+ \left(\tfrac{40}{3}R_2 - 10R_2^3 - 4R_2^5\right)a_{26}H_2^7 - 8R^3a_{41}H_2^4 + 4R_2^3a_{42}H_2^5\Big] \\
&+ a_{24}\left\{\left(16R_2 + 6R_2^3 - \tfrac{12}{5}R_2^5\right)a_{25}H_2^7 + \left(\tfrac{120}{7}R_2 - 8R_2^5\right)a_{26}H_2^8\right. \\
&\left. - 12R_2^3a_{41}H_2^5\right\} + a_{25}\left\{\left(20R_2 + 20R_2^3 + 12R_2^5 + \tfrac{20}{7}R_2^7\right)a_{26}H_2^9\right. \\
&\left. - \left(10R_2^3 + 12R_2^5\right)a_{42}H_2^7 - \left(24R_2^3 - \tfrac{24}{5}R_2^5\right)a_{41}H_2^6\right\} \\
&+ a_{26}\left\{-\left(16R_2^3 + 16R_2^5\right)a_{42}H_2^8 - \left(30R_2^3 - 12R_2^5\right)a_{41}H_2^7\right\} \\
&+ 12R_2^3a_{41}a_{42}H_2^5\Big]
\end{aligned} \tag{A3}$$

The strain energy rate for the total cross section slice of unit thickness through the model is then:

$$\dot{E}_{\epsilon\gamma} = \dot{E}_{\epsilon\gamma_1} + \dot{E}_{\epsilon\gamma_2} \tag{A4}$$

The rate of change of potential energy for the unit slice follows upon integration of the expression:

$$\dot{e}_{pot} = \rho_i g v_i \partial x \partial y \qquad \text{(see also p. 239).} \tag{A5}$$

across layer 1 when i is 1 and layer 2 when $i = 2$. Here the vertical velocity component is derived from the appropriate stream functions (34) or (35). This procedure leads to the potential energy rate for the whole unit section across the model:

$$\begin{aligned}
\dot{E}_{pot} = \dot{E}_{pot1} + \dot{E}_{pot2} = \rho_1 g\Big[&-\left(\tfrac{1}{3}H_1^4a + \tfrac{1}{4}H_1^5b + \tfrac{1}{5}H_1^6c + \tfrac{1}{6}H_1^7d\right)R_1 \\
&+ \left(\tfrac{1}{3}H_1^6c + \tfrac{5}{12}H_1^7d\right)R_1^3\Big] + \rho_2 g\Big[\left(H_2^2a_{21} + \tfrac{1}{2}H_2^3a_{22} + \tfrac{1}{3}H_2^4a_{23}\right. \\
&+ \tfrac{1}{4}H_2^5a_{24} + \tfrac{1}{5}H_2^6a_{25} + \tfrac{1}{6}H_2^7a_{26}\Big)R_2 + \left(H_2^4a_{41} + \tfrac{1}{2}H_2^5a_{42}\right. \\
&\left. - \tfrac{1}{3}H_2^6a_{25} - \tfrac{5}{12}H_2^7a_{26}\right)R_2^3\Big]
\end{aligned} \tag{A6}$$

R_i ($i = 1$ to 2) is the aspect ratio, L/H_i of the layers.

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```
10   !"COMB86NAPW"
20 !A plotting,printing program for double-layer spreading nappe with vanishing
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80 !The two polynomial STREAM FUNCTIONS are:
90 !Psi1=-(AY^2+BY^3+CY^4+DY^5+EY^6+FY^7)X+(Cy^2+5/3Dy^3+10/3Ey^4+14/3Fv^5)x^3-(
Ey^2+7/3Fy^3)x^5
100 !Psi2=(A21+A22y+A23y^2+A24y^3+A25y^4+A26y^5)x+(A41+A42y-(A25+5*A61)y^2-5/3*(
A26+A62)y^3)x^3+(A61+A62y)x^5
110 !!
120 !H1,H2 are thicknesses,Rh=H1/H2,L= length,R2(INPUT item!)=R=L/H2,R1=L/H1,Ro1
,Ro2 are densities,Rro=Ro1/Ro2,Mu1,Mu2 are viscosities,Rm(INPUT item')=Mu1/Mu2
130 !!
140! Gr=acceleration of gravity, U1,V1,U2,V2 are horizontal (U) and vertical (V)
 velocity components in layer 1 and 2
150 !!
160 !Ufree,Vfree are velocitis at front face of free-slip model;X1,Y1;X2,Y2 are
position vectors in layer 1,resp.layer 2;Mi=X/L,Ni=Y/H in lines 1470,1490 ff.
170 ! are normalized position vectors
180 !!
190 ! If hight is in cm, density in g/cm^3, Gr in cm/s^2  and viscosity in pois
then velocity is in cm/year;with hight in m, density in kg/m^3, viscosity in
200 !Pascal s and Gr in m/s^2 then velocity is in meter per year
210 !!
220  PRINTER IS 705 !(705 is really a plotter:HP7470A)
230  PRINT "IN;IP,VS2;"!This prepares for PLOTTING, and specifies PEN VELOCITY
240  TRACE PAUSE
```

```
250   !When "PRINT SC---,---,---,---"etc. appears on screen press "EDIT, 280" fol
lowed by "EXECUTE", and line 280 will appear
260 !If scale values are inserted prior to start then press "CONTINUE" directly
when line 280 appears,else:
270 !Insert desired scale values, (considering the R2,Sym and Rm commands to be
INPUT later!) press "ENTER" followed by "RUN"
280   PRINT "SC-50,50,0,72"!This is an example
290 !!
300 !After "RUN" is pressed,"PRINT "SC--,--,--,--" again appears on screen, and
now with numbers inserted
310 !!
320 !Then press "CONTINUE" which puts"R2,Rm,Ini,Sym,Bou,Lay1,Lay2,Print,etc" on
screen; insert values for R2,Rm,etc. and press "CONTINUE" anew to start plotting
330   !!
340   ! Together with the input values for R2 and Rm the following "yes/no" or 1/
zero choices are possible:
350   !Ini =0 plots initial profile,Ini=1 plots deformed profile,Sym=-1 plots sym
metric profile,Sym=0 plots right hand half of profile
360   !!
370   !Bou=1 plots top boundary first,Bou=0 plots base first; Lay1=1,Lay2=0,plots
 layer 1 only,Lay2=1,Lay1=0 plots only layer 2,Lay1=1,Lay2=1 plots both layers
380   !!
390   !!Printing of velocities follows plotting if Print is put =1 and Velfro and
Veltop are given values 0 or 1:Print=1 and Velfro=1,Veltop=0 gives the velocity
400 !at the front face of both layers;Print=1,Velfro=0,Veltop=1 gives the veloci
ty at the top boundary of both layers;Print=1 and Velfro=0,Veltop=0 prints the
410 !velocity throughout bouth layers. Print=0 prevents velocity to be displayed
 or printed; If Lay1=Lay2 =0, Print=1 and Velfro=0 or 1,Veltop=1 or 0 then
420 !velocity is printed without activating plotting
430   !!
440 !For plotting of both upper and lower boundaries press "RUN" and "CONTINUE"
directly after the first plotting or printing ends,then  make input for "R2,Rm,
450 !Ini etc", but now using the alternative command for Bou,and press CONTINUE
again
460   INPUT "R2,Rm,Ini,Sym,Bou,Lay1,Lay2,",R2,Rm,Ini,Sym,Bou,Lay1,Lay2
470   INPUT "Print,Velfro,Veltop",Print,Velfro,Veltop
480    COM REAL H1,H2,R,R2,R1,Rh,Gr,Ro1,Ro2,Rro,Mu1,Mu2,Rm,Den,Epota,Epotb,Estra,
Estrb,Estrab
490    H1=5*10^3!cm,change as needed befor start
500   H2=5*10^5!cm,change as nedded --- "" ---
510    Rh=H1/H2
520    R=R2
530    L=R*H2
540    R1=R2/Rh
550    Gr=981!cm/s^2
560    Ro1=2.8!g/cm^3,change as needed befor start
570    Ro2=2.8!g/cm^3,change as needed --- "" ---
580    Mu2=10^22 !Pois,change as needed --- "" ---
590    Mu1=Mu2*Rm
600    Rro=Ro1/Ro2
610    CALL Energyfactors
620    !!
630   !CALCULATION OF THE COEFFICIENTS A,B,...,A41,A42 IN THE STREAM FUNCTIONS
Psi(1) AND Psi(2) FOLLOW
640    Q=Epotb/Epota
650    Phi=(2*Q*Estra-Estrab)/(2*Estrb*H2-Q*Estrab*H2)
660    A=-(Epota+Epotb*H2*Phi)/(Estra+Estrb*H2^2*Phi^2+Estrab*H2*Phi)
670    B=Phi*A
680    C=(A+(3+3/Rh)*H1*B)/(Den*H1^2) !"Den" is defined line 1760
690    D=-(A+(3+3/Rh)*H1*B)/(Den*(5+5/Rh)*H1^3)
700    A21=-(A*H1^2+B*H1^3+C*H1^4+D*H1^5)
710    A22=-(2*A*H1+3*B*H1^2+4*C*H1^3+5*D*H1^4)
720    A23=-Rm*(A+3*B*H1+(9-3/Rm)*C*H1^2+(15-5/Rm)*D*H1^3)
730    A24=-Rm*B+(2*Rm-6)*C*H1+(5*Rm-15)*D*H1^2
740    A25=-Rm*(C+5*D*H1)
750    A26=-Rm*D
760    A41=C*H1^2+5/3*D*H1^3
770    A42=2*C*H1+5*D*H1^2
780   IF Lay1=0 THEN 1100
790   IF Ini=0 THEN
800   PRINT "LT2,1" !Commands plot with dashed line (for initial profile)
810   ELSE
820   PRINT "LT" !Commands plot with solid line (for deformed profile)
830   END IF
840    FOR M=Sym TO 1 STEP .1  !Sym is input as 0 or -1,Sym=0 plots right hand
half profile,Sym=-1 plots full symmetric profile
850    IF M=Sym THEN
860    FOR M1=10*R2*Sym TO 10*R2 STEP R2/10 !This draws the top (If Bou=1) or b
tom (If Bou=0) boundary of layer 1
870    X1=M1*H1*R1/(10*R2)
880    Y1=H1*Bou
890    U1=((2*A*Y1+3*B*Y1^2+4*C*Y1^3+5*D*Y1^4)*X1-(2*C*Y1+5*D*Y1^2)*X1^3)*3.1*1
7*Ini
900    V1=(-(A*Y1^2+B*Y1^3+C*Y1^4+D*Y1^5)+(3*C*Y1^2+5*D*Y1^3)*X1^2)*3.1*10^7*In
910    X=M1+U1*10
920    Y=(10*Rh+V1*10)*Bou !H2 is taken to equal 10 USERS UNITS so that H1= 10*
  USERS UNITS; the velocities are multiplied by a 10-units time periode
930    PRINT "PA",X,",",Y,";PD;SP1;"
940    NEXT M1
950    PRINT "PU;"
960    END IF
970    FOR N1=0 TO 10 STEP .2 !This draws the initially straight vertical marke
 in layer 1 Because of the small Rh value the vertical scale must generally be
980 ! greatly exaggerated when plotting structures in layer 1
990    X1=R1*H1*M
1000   Y1=N1*H1/10
1010   U1=((2*A*Y1+3*B*Y1^2+4*C*Y1^3+5*D*Y1^4)*X1-(2*C*Y1+5*D*Y1^2)*X1^3)*3.1*
7*Ini
1020   V1=(-(A*Y1^2+B*Y1^3+C*Y1^4+D*Y1^5)+(3*C*Y1^2+5*D*Y1^3)*X1^2)*3.1*10^7*I
1030   X=10*R*M+U1*10
1040   Y=N1*Rh+V1*10 !See 920
1050   PRINT "PA",X,",",Y,";PD;SP1;"
1060   NEXT N1
1070   PRINT "PU;"
1080   NEXT M
1090   PRINT "PU;"
1100   IF Lay2=0 THEN 1420
1110   IF Ini=0 THEN
1120   PRINT "SP1;LT2,1"!See 800
1130   ELSE
1140   PRINT "SP2;LT"!See 820
1150   END IF
1160   FOR M=Sym TO 1 STEP .1 !Here starts preparation for  plotting of layer
1170   IF M=Sym THEN
1180   FOR M2=10*R2*Sym TO 10*R2 STEP R2/10
1190   X2=M2*H2*R2/(10*R2)
1200   Y2=H2*Bou
1210   U2=(-(A22+2*A23*Y2+3*A24*Y2^2+4*A25*Y2^3+5*A26*Y2^4)*X2+(-A42+2*A25*Y2+
26*Y2^2)*X2^3)*3.1*10^7*Ini
1220   V2=(A21+A22*Y2+A23*Y2^2+A24*Y2^3+A25*Y2^4+A26*Y2^5+(3*A41+3*A42*Y2-3*A
2^2-5*A26*Y2^3)*X2^2)*3.1*10^7*Ini
1230   X=M2+10*U2
1240   Y=(10+10*V2)*Bou+10*Rh ! H2 is taken to equal 10 USERS UNITS and the ve
ity is multiplied by a periode of 10 time units
1250   PRINT "PA",X,",",Y,";PD;"
1260   NEXT M2
1270   PRINT "PU;"
1280   END IF
1290   FOR N2=0 TO 10 STEP .5 !Now follow plotting of initially straight vert
markers in layer 2
1300   X2=R2*H2*M
1310   Y2=N2*H2/10
1320   U2=(-(A22+2*A23*Y2+3*A24*Y2^2+4*A25*Y2^3+5*A26*Y2^4)*X2+(-A42+2*A25*Y2
26*Y2^2)*X2^3)*3.1*10^7*Ini
1330   V2=(A21+A22*Y2+A23*Y2^2+A24*Y2^3+A25*Y2^4+A26*Y2^5+(3*A41+3*A42*Y2-3*A
2^2-5*A26*Y2^3)*X2^2)*3.1*10^7*Ini
1340   X=10*R2*M+10*U2
1350   Y=10*Rh+N2+10*V2 !SEE 1240
1360   PRINT "PA",X,",",Y,";PD;"
1370   NEXT N2
1380   PRINT "PU;"
1390   NEXT M
1400   PRINT "PU;SP0;"!Here plotting ends and printing of velocities etc star
 according to the commands below
1410   !!
1420   IF Print=0 THEN 1710
1430   Epsi=Ro2*Gr*H2/(8*Mu2)
```

```
)  !When "PRINT SC---,---,---,---"etc. appears on screen press "EDIT, 280" fol
ued by "EXECUTE", and line 280 will appear
 !If scale values are inserted prior to start then press "CONTINUE" directly
n line 280 appears,else:
 !Insert desired scale values, (considering the R2,Sym and Rm commands to be
UT later!) press "ENTER" followed by "RUN"
  PRINT "SC-50,50,0,72"!This is an example
 !!
 !After "RUN" is pressed,"PRINT "SC--,--,--,--" again appears on screen, and
 with numbers inserted
 !!
 !Then press "CONTINUE" which puts"R2,Rm,Ini,Sym,Bou,Lay1,Lay2,Print,etc" on
een; insert values for R2,Rm,etc. and press "CONTINUE" anew to start plotting
 !!
 ! Together with the input values for R2 and Rm the following "yes/no" or 1/
 choices are possible:
 !Ini =0 plots initial profile,Ini=1 plots deformed profile,Sym=-1 plots sym
ic profile,Sym=0 plots right hand half of profile
 !!
 !Bou=1 plots top boundary first,Bou=0 plots base first; Lay1=1,Lay2=0,plots
er 1 only,Lay2=1,Lay1=0 plots only layer 2,Lay1=1,Lay2=1 plots both layers
 !!
 !!Printing of velocities follows plotting if Print is put =1 and Velfro and
op are given values,0 or 1; Print=1 and Velfro=1,Veltop=0 gives the velocity
at the front face of both layers;Print=1,Velfro=0,Veltop=1 gives the veloci
 the top boundary of both layers;Print=1 and Velfro=0,Veltop=0 prints the
velocity throughout bouth layers. Print=0 prevents velocity to be displayed
rinted; If Lay1=Lay2 =0, Print=1 and Velfro=0 or 1,Veltop=1 or 0 then
velocity is printed without activating plotting
 !!
For plotting of both upper and lower boundaries press "RUN" and "CONTINUE"
:ly after the first plotting or printing ends,then  make input for "R2,Rm,
Ini etc", but now using the alternative command for Bou,and press CONTINUE

NPUT "R2,Rm,Ini,Sym,Bou,Lay1,Lay2,",R2,Rm,Ini,Sym,Bou,Lay1,Lay2
NPUT "Print,Velfro,Veltop",Print,Velfro,Veltop
COM REAL H1,H2,R,R2,R1,Rh,Gr,Ro1,Ro2,Rro,Mu1,Mu2,Rm,Den,Epota,Epotb,Estra,
Estrab
H1=5*10^3!cm,change as needed befor start
H2=5*10^5!cm,change as nedded --- "" ---
Rh=H1/H2
R=R2
=R*H2
R1=R2/Rh
Gr=981!cm/s^2
Ro1=2.8!g/cm^3,change as needed befor start
Ro2=2.8!g/cm^3,change as needed --- "" ---
Mu2=10^22 !Pois,change as needed --- "" ---
Mu1=Mu2*Rm
Rro=Ro1/Ro2
CALL Energyfactors
!
CALCULATION OF THE COEFFICIENTS A,B,...,A41,A42 IN THE STREAM FUNCTIONS
AND Psi(2) FOLLOW
=Epotb/Epota
Ini=(2*Q*Estra-Estrab)/(2*Estrb*H2-Q*Estrab*H2)
=-(Epota+Epotb*H2*Phi)/(Estra+Estrb*H2^2*Phi^2+Estrab*H2*Phi)
Phi*A
(A+(3+3/Rh)*H1*B)/(Den*H1^2) !"Den" is defined line 1760
-(A+(3+3/Rh)*H1*B)/(Den*(5+5/Rh)*H1^3)
1=-(A*H1^2+B*H1^3+C*H1^4+D*H1^5)
2=-(2*A*H1+3*B*H1^2+4*C*H1^3+5*D*H1^4)
3=-Rm*(A+3*B*H1+(9-3/Rm)*C*H1^2+(15-5/Rm)*D*H1^3)
4=-Rm*B+(2*Rm-6)*C*H1+(5*Rm-15)*D*H1^2
5=-Rm*(C+5*D*H1)
6=-Rm*D
=C*H1^2+5/3*D*H1^3
5=2*C*H1+5*D*H1^2
ay1=0 THEN 1100
ni=0 THEN
T "LT2,1" !Commands plot with dashed line (for initial profile)

T "LT" !Commands plot with solid line (for deformed profile)
IF
```

```
840   FOR M=Sym TO 1 STEP .1  !Sym is input as 0 or -1.Sym=0 plots right hand
half profile,Sym=-1 plots full symmetric profile
850   IF M=Sym THEN
860   FOR M1=10*R2*Sym TO 10*R2 STEP R2/10 !This draws the top (If Bou=1) or bot
tom (If Bou=0) boundary of layer 1
870   X1=M1*H1*R1/(10*R2)
880   Y1=H1*Bou
890   U1=((2*A*Y1+3*B*Y1^2+4*C*Y1^3+5*D*Y1^4)*X1-(2*C*Y1+5*D*Y1^2)*X1^3)*3.1*10^
7*Ini
900   V1=(-(A*Y1^2+B*Y1^3+C*Y1^4+D*Y1^5)+(3*C*Y1^2+5*D*Y1^3)*X1^2)*3.1*10^7*Ini
910   X=M1+U1*10
920   Y=(10*Rh+V1*10)*Bou !H2 is taken to equal 10 USERS UNITS so that H1= 10*Rh
 USERS UNITS; the velocities are multiplied by a 10-units time periode
930   PRINT "PA",X,",",Y,";PD;SP1;"
940   NEXT M1
950   PRINT "PU;"
960   END IF
970   FOR N1=0 TO 10 STEP .2 !This draws the initially straight vertical markers
 in layer 1 Because of the small Rh value the vertical scale must generally be
980 ! greatly exaggerated when plotting structures in laver 1
990   X1=R1*H1*M
1000  Y1=N1*H1/10
1010  U1=((2*A*Y1+3*B*Y1^2+4*C*Y1^3+5*D*Y1^4)*X1-(2*C*Y1+5*D*Y1^2)*X1^3)*3.1*10^
7*Ini
1020  V1=(-(A*Y1^2+B*Y1^3+C*Y1^4+D*Y1^5)+(3*C*Y1^2+5*D*Y1^3)*X1^2)*3.1*10^7*Ini
1030  X=10*R*M+U1*10
1040  Y=N1*Rh+V1*10 !See 920
1050  PRINT "PA",X,",",Y,";PD;SP1;"
1060  NEXT N1
1070  PRINT "PU;"
1080  NEXT M
1090  PRINT "PU;"
1100  IF Lay2=0 THEN 1420
1110  IF Ini=0 THEN
1120  PRINT "SP1;LT2,1"!See 800
1130  ELSE
1140  PRINT "SP2;LT"!See 820
1150  END IF
1160  FOR M=Sym TO 1 STEP .1 !Here starts preparation for  plotting of laver 2
1170  IF M=Sym THEN
1180  FOR M2=10*R2*Sym TO 10*R2 STEP R2/10
1190  X2=M2*H2*R2/(10*R2)
1200  Y2=H2*Bou
1210  U2=(-(A22+2*A23*Y2+3*A24*Y2^2+4*A25*Y2^3+5*A26*Y2^4)*X2+(-A42+2*A25*Y2+5*A
26*Y2^2)*X2^3)*3.1*10^7*Ini
1220  V2=(A21+A22*Y2+A23*Y2^2+A24*Y2^3+A25*Y2^4+A26*Y2^5+(3*A41+3*A42*Y2-3*A25*Y
2^2-5*A26*Y2^3)*X2^2)*3.1*10^7*Ini
1230  X=M2+10*U2
1240  Y=(10+10*V2)*Bou+10*Rh ! H2 is taken to equal 10 USERS UNITS and the veloc
ity is multiplied by a periode of 10 time units
1250  PRINT "PA",X,",",Y,";PD;"
1260  NEXT M2
1270  PRINT "PU;"
1280  END IF
1290  FOR N2=0 TO 10 STEP .5 !Now follow plotting of initially straight vertical
markers in layer 2
1300  X2=R2*H2*M
1310  Y2=N2*H2/10
1320  U2=(-(A22+2*A23*Y2+3*A24*Y2^2+4*A25*Y2^3+5*A26*Y2^4)*X2+(-A42+2*A25*Y2+5*A
26*Y2^2)*X2^3)*3.1*10^7*Ini
1330  V2=(A21+A22*Y2+A23*Y2^2+A24*Y2^3+A25*Y2^4+A26*Y2^5+(3*A41+3*A42*Y2-3*A25*Y
2^2-5*A26*Y2^3)*X2^2)*3.1*10^7*Ini
1340  X=10*R2*M+10*U2
1350  Y=10*Rh+N2+10*V2 !SEE 1240
1360  PRINT "PA",X,",",Y,";PD;"
1370  NEXT N2
1380  PRINT "PU;"
1390  NEXT M
1400  PRINT "PU;SP0;"!Here plotting ends and printing of velocities etc starts
 according to the commands below
1410  !!
1420  IF Print=0 THEN 1710
1430  Epsi=Ro2*Gr*H2/(8*Mu2)
```

```
1440  Ufree=Epsi*H2*R2*3.1*10^7
1450  Vfree=-Epsi*H2*3.1*10^7
1460  IF Velfro=1 THEN 1490
1470  FOR Mi=1 TO 0 STEP -.1
1480  IF Veltop=1 THEN 1510
1490  FOR Ni=1 TO 0 STEP -.1
1500  IF Velfro=1 THEN Mi=1
1510  X1=Mi*R1*H1
1520  X2=Mi*R2*H2
1530 IF Veltop=1 THEN Ni=1
1540  Y1=Ni*H1
1550  Y2=Ni*H2
1560  U1=((2*A*Y1+3*B*Y1^2+4*C*Y1^3+5*D*Y1^4)*X1-(2*C*Y1+5*D*Y1^2)*X1^3)*3.1*10^
7
1570  V1=(-(A*Y1^2+B*Y1^3+C*Y1^4+D*Y1^5)+(3*C*Y1^2+5*D*Y1^3)*X1^2)*3.1*10^7
1580  U2=(-(A22+2*A23*Y2+3*A24*Y2^2+4*A25*Y2^3+5*A26*Y2^4)*X2+(-A42+2*A25*Y2+5*A
26*Y2^2)*X2^3)*3.1*10^7
1590  V2=(A21+A22*Y2+A23*Y2^2+A24*Y2^3+A25*Y2^4+A26*Y2^5+(3*A41+3*A42*Y2-3*A25*Y
2^2-5*A26*Y2^3)*X2^2)*3.1*10^7
1600  PRINTER IS 1
1610  IF Mi=Ni THEN PRINT "COMB86NAPW::H2=";H2;"H1=";H1;"R2=";R2;"Mu2=";Mu2;"Mu1
=";Mu1;"Ro1=";Ro1;"Ro2=";Ro2;"Ufree=";Ufree;"Vfree=";Vfree
1620  IF Mi=1 THEN
1630  IF Ni=1 THEN PRINT USING "5X,12A,14A,14A,8A,2A,XX,2A";"U1","U2","V1","V2",
"Mi","Ni"
1640  END IF
1650  PRINT TAB(54);Mi;Ni
1660 PRINT USING "DD.10D";U1,U2,V1,V2
1670  NEXT Ni
1680  NEXT Mi
1690  PRINT "      "
1700  PRINT "      "
1710  END
1720  ! ++++++++++++++++++++++++++++++
1730  SUB Energyfactors
1740  !Den IS TERM IN FOLLOWING EXPRESSIONS ,SEE ALSO C AND D,LINES 680,690
1750  COM REAL H1,H2,R,R2,R1,Rh,Gr,Ro1,Ro2,Rro,Mu1,Mu2,Rm,Den,Epota,Epotb,Estra,
Estrb,Estrab
1760  Den=6/Rh-9-9/Rh^2-24/(Rh*Rm)+(3+3/Rh^3-3/Rh+9/Rh^2+12/(Rh*Rm))/(1+1/Rh)
1770  Ca=1/Den
1780  Cb=(3+3/Rh)/Den
1790  Da=-Ca/(5*(1+1/Rh))
1800  Db=-Cb/(5*(1+1/Rh))
1810  A21a=-Rh^2*(1+Ca+Da)
1820  A21b=-Rh^3*(1+Cb+Db)
1830  A22a=-Rh*(2+4*Ca+5*Da)
1840  A22b=-Rh^2*(3+4*Cb+5*Db)
1850  A23a=-Rm+(3-9*Rm)*Ca+(5-15*Rm)*Da
1860  A23b=-3*Rm*Rh+(3-9*Rm)*Cb*Rh+(5-15*Rm)*Db*Rh
1870  A24a=(2*Rm-6)*Ca/Rh+(5*Rm-15)*Da/Rh
1880  A24b=-Rm+(2*Rm-6)*Cb+(5*Rm-15)*Db
1890  A25a=-Rm*(Ca+5*Da)/Rh^2
1900  A25b=-Rm*(Cb+5*Db)/Rh
1910  A26a=-Rm*Da/Rh^3
1920  A26b=-Rm*Db/Rh^2
1930  A41a=Ca+5/3*Da
1940  A41b=(Cb+5/3*Db)*Rh
1950  A42a=(2*Ca+5*Da)/Rh
1960  A42b=2*Cb+5*Db
1970  C1=64/7*R1+132/15*R1^3+72/15*R1^5+4/7*R1^7
1980  D1=100/9*R1+100/7*R1^3+12*R1^5+100/21*R1^7
1990  Ac=(64/5*R1-24/9*R1^3-8/5*R1^5)
2000  Ad=(40/3*R1-10*R1^3-4*R1^5)
2010  Bc=(16*R1+6*R1^3-12/5*R1^5)
2020  Bd=(120/7*R1-8*R1^5)
2030  Cd=20*R1+20*R1^3+12*R1^5+20*R1^7
2040  !Estra1,Estrb1 AND Estrab1 BELOW ARE FACTORS TO MULTIPLY WITH  A^2*H2^4,B^
2*H2^6 AND A*B*H2^5 IN STRAIN ENERGY EXPRESSION  FOR LAYER 1
2050  !Epota1,Epotb1 ARE FACTORS TO MULTIPLY WITH A*H2^4 AND B*H2^5 IN POTENTIAL
 ENERGY EXPRESSION FOR LAYER 1
2060  Estra1=(16/3*R1+4/3*R1^3+C1*Ca^2+D1*Da^2+Ac*Ca+Ad*Da+Cd*Ca*Da)*Rh^4
2070  Estrb1=(36/5*R1+4*R1^3+C1*Cb^2+D1*Db^2+Bc*Cb+Bd*Db+Cd*Cb*Db)*Rh^6
2080  Estrab1=(12*R1+4*R1^3+2*C1*Ca*Cb+2*D1*Da*Db+Ac*Cb+Ad*Db+Bc*Ca+Bd*Da+Cd*Ca*
Db+Cd*Cb*Da)*Rh^5
2090  Epota1=(-(1/3+1/5*Ca+1/6*Da)*R1+(1/3*Ca+5/12*Da)*R1^3)*Rh^4
2100  Epotb1=(-(1/4+1/5*Cb+1/6*Db)*R1+(1/3*Cb+5/12*Db)*R1^3)*Rh^5
2110  !HERE  FOLLOW  CALCULATIONS  OF THE COEFFICIENTS IN THE ENERGY EXPI
FOR LAYER 2
2120  Q1=16/3*R+4/3*R^3
2130  Q2=36/5*R+4*R^3
2140  Q3=64/7*R+44/5*R^3+24/5*R^5+4/7*R^7
2150  Q4=100/9*R+100/7*R^3+12*R^5+100/21*R^7
2160  Q5=4*R^3+36/5*R^5
2170  Q6=8*(R-R^3)
2180  Q7=8*(R-5/3*R^3)
2190  Q8=12*R+4*R^3
2200  Q9=64/5*R-8/3*R^3-8/5*R^5
2210  Q10=40/3*R-10*R^3-4*R^5
2220  Q11=16*R+6*R^3-12/5*R^5
2230  Q12=120/7*R-8*R^5
2240  Q13=20*(R+R^3+3/5*R^5+1/7*R^7)
2250  Q14=-10*R^3-12*R^5
2260  Q15=-24*R^3+24/5*R^5
2270  Q16=-16*(R^3+R^5)
2280  Q17=-30*R^3+12*R^5
2290  !Estra2,Estrb2= AND Estrab2 ARE FACTORS TO MULTIPLY WITH A^2*H2^4,
 AND A*B*H2^5 IN THE STRAIN ENERGY EXPRESSION FOR LAYER 2
2300  !!!
2310  Estra12=4*R*A22a^2+Q1*A23a^2+Q2*A24a^2+Q3*A25a^2+Q4*A26a^2+12*R^3+
5*A42a^2+A22a*8*(A23a*R+A24a*R+Q6/8*A25a+Q7/8*A26a+A42a*R^3)
2320  Estra22=A23a*(Q8*A24a+Q9*A25a+Q10*A26a-8*R^3*A41a+4*R^3*A42a)+A24.
5a+Q12*A26a-12*R^3*A41a)+A25a*(Q13*A26a+Q14*A42a+Q15*A41a)+12*R^3*A41a*
2330  Estra2=Estra12+Estra22+A26a*(Q16*A42a+Q17*A41a)
2340  !!!
2350  Estrb12=4*R*A22b^2+Q1*A23b^2+Q2*A24b^2+Q3*A25b^2+Q4*A26b^2+12*R^3
5*A42b^2+A22b*8*(A23b*R+A24b*R+Q6/8*A25b+Q7/8*A26b+A42b*R^3)
2360  Estrb22=A23b*(Q8*A24b+Q9*A25b+Q10*A26b-8*R^3*A41b+4*R^3*A42b)+A24
5b+Q12*A26b-12*R^3*A41b)+A25b*(Q13*A26b+Q14*A42b+Q15*A41b)+12*R^3*A41b*
2370  Estrb2=Estrb12+Estrb22+A26b*(Q16*A42b+Q17*A41b)
2380  !!!
2390  Estrab12=8*R*A22a*A22b+2*Q1*A23a*A23b+2*Q2*A24a*A24b+2*Q3*A25a*A2
26a*A26b+24*R^3*A41a*A41b+2*Q5*A42a*A42b+A22a*(8*A23b*R+8*A24b*R+Q6*A25
2400  Estrab22=A22a*(Q7*A26b+8*R^3*A42b)+A22b*8*(A23a*R+A24a*R+Q6/8*A25
6a+R^3*A42a)+A23a*(Q8*A24b+Q9*A25b+Q10*A26b-8*R^3*A41b+4*R^3*A42b)
2410  Estrab32=A23b*(Q8*A24a+Q9*A25a+Q10*A26a-8*R^3*A41a+4*R^3*A42a)+A2
25b+Q12*A26b-12*R^3*A41b)+A24b*(Q11*A25a+Q12*A26a-12*R^3*A41a)
2420  Estrab42=A25a*(Q13*A26b+Q14*A42b+Q15*A41b)+A25b*(Q13*A26a+Q14*A42
a)+A26a*(Q16*A42b+Q17*A41b)+A26b*(Q16*A42a+Q17*A41a)+12*R^3*A41a*A42b
2430  Estrab2=Estrab12+Estrab22+Estrab32+Estrab42+12*R^3*A41b*A42a
2440  !!!
2450  Estra=Mu1*Estra1+Mu2*Estra2
2460  Estrb=Mu1*Estrb1+Mu2*Estrb2
2470  Estrab=Mu1*Estrab1+Mu2*Estrab2
2480  !!!
2490  !BELOW FOLLOW CALCULATION OF THE FACTORS TO MULTIPLY WITH A*H2^4
5 IN THE EXPRESSION FOR POTENTIAL ENERGY IN LAYER 2
2500  Epota2=(A21a+.5*A22a+1/3*A23a+1/4*A24a+1/5*A25a+1/6*A26a)*R+(A41.
-1/3*A25a-5/12*A26a)*R^3
2510  Epotb2=(A21b+.5*A22b+1/3*A23b+1/4*A24b+1/5*A25b+1/6*A26b)*R+(A41
-1/3*A25b-5/12*A26b)*R^3
2520  !!!
2530  Epota=Ro2*Gr*(Epota1*Rro+Epota2)
2540  Epotb=Ro2*Gr*(Epotb1*Rro+Epotb2)
2550  SUBEND
```


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```
0  Ufree=Epsi*H2*R2*3.1*10^7
0  Vfree=-Epsi*H2*3.1*10^7
0  IF Velfro=1 THEN 1490
0  FOR Mi=1 TO 0 STEP -.1
)  IF Veltop=1 THEN 1510
)  FOR Ni=1 TO 0 STEP -.1
)  IF Velfro=1 THEN Mi=1
)  X1=Mi*R1*H1
)  X2=Mi*R2*H2
) IF Veltop=1 THEN Ni=1
   Y1=Ni*H1
   Y2=Ni*H2
   U1=((2*A*Y1+3*B*Y1^2+4*C*Y1^3+5*D*Y1^4)*X1-(2*C*Y1+5*D*Y1^2)*X1^3)*3.1*10^

   V1=(-(A*Y1^2+B*Y1^3+C*Y1^4+D*Y1^5)+(3*C*Y1^2+5*D*Y1^3)*X1^2)*3.1*10^7
   U2=(-(A22+2*A23*Y2+3*A24*Y2^2+4*A25*Y2^3+5*A26*Y2^4)*X2^3)*3.1*10^7
2^2)*X2^3)*3.1*10^7
   V2=(A21+A22*Y2+A23*Y2^2+A24*Y2^3+A25*Y2^4+A26*Y2^5+(3*A41+3*A42*Y2-3*A25*Y
5*A26*Y2^3)*X2^2)*3.1*10^7
   PRINTER IS 1
   IF Mi=Ni THEN PRINT "COMB86NAPW::H2=";H2;"H1=";H1;"R2=";R2;"Mu2=";Mu2;"Mu1
.1;"Ro1=";Ro1;"Ro2=";Ro2;"Ufree=";Ufree;"Vfree=";Vfree
   IF Mi=1 THEN
   IF Ni=1 THEN PRINT USING "5X,12A,14A,14A,8A,2A,XX,2A";"U1","U2","V1","V2",
"Ni"
   END IF
   PRINT TAB(54);Mi;Ni
PRINT USING "DD.10D";U1,U2,V1,V2
 NEXT Ni
 NEXT Mi
 PRINT "      "
 PRINT "      "
END
! ++++++++++++++++++++++++++++++
SUB Energyfactors
!Den IS TERM IN FOLLOWING EXPRESSIONS ,SEE ALSO C AND D,LINES 680,690
COM REAL H1,H2,R,R2,R1,Rh,Gr,Ro1,Ro2,Rro,Mu1,Mu2,Rm,Den,Epota,Epotb,Estra,
Estrab
Den=6/Rh-9-9/Rh^2-24/(Rh*Rm)+(3+3/Rh^3-3/Rh+9/Rh^2+12/(Rh*Rm))/(1+1/Rh)
Ca=1/Den
Cb=(3+3/Rh)/Den
Da=-Ca/(5*(1+1/Rh))
Db=-Cb/(5*(1+1/Rh))
A21a=-Rh^2*(1+Ca+Da)
A21b=-Rh^3*(1+Cb+Db)
A22a=-Rh*(2+4*Ca+5*Da)
A22b=-Rh^2*(3+4*Cb+5*Db)
A23a=-Rm+(3-9*Rm)*Ca+(5-15*Rm)*Da
A23b=-3*Rm*Rh+(3-9*Rm)*Cb*Rh+(5-15*Rm)*Db*Rh
A24a=(2*Rm-6)*Ca/Rh+(5*Rm-15)*Da/Rh
A24b=-Rm+(2*Rm-6)*Cb+(5*Rm-15)*Db
A25a=-Rm*(Ca+5*Da)/Rh^2
A25b=-Rm*(Cb+5*Db)/Rh
A26a=-Rm*Da/Rh^3
A26b=-Rm*Db/Rh^2
A41a=Ca+5/3*Da
A41b=(Cb+5/3*Db)*Rh
A42a=(2*Ca+5*Da)/Rh
A42b=2*Cb+5*Db
Q1=64/7*R1+132/15*R1^3+72/15*R1^5+4/7*R1^7
Q2=100/9*R1+100/7*R1^3+12*R1^5+100/21*R1^7
Q3=(64/5*R1-24/9*R1^3-8/5*R1^5)
Q4=(40/3*R1-10*R1^3-4*R1^5)
Q5=(16*R1+6*R1^3-12/5*R1^5)
Q6=(120/7*R1-8*R1^5)
Q7=20*R1+20*R1^3+12*R1^5+20*R1^7
!Estra1,Estrb1 AND Estrab1 BELOW ARE FACTORS TO MULTIPLY WITH  A^2*H2^4,B^
2 AND A*B*H2^5 IN STRAIN ENERGY EXPRESSION  FOR LAYER 1
!Epota1,Epotb1 ARE FACTORS TO MULTIPLY WITH A*H2^4 AND B*H2^5 IN POTENTIAL
EXPRESSION FOR LAYER 1
Epota1=(16/3*R1+4/3*R1^3+C1*Ca^2+D1*Da^2+Ac*Ca+Ad*Da+Cd*Ca*Da)*Rh^4
Epotb1=(36/5*R1+4*R1^3+C1*Cb^2+D1*Db^2+Bc*Cb+Bd*Db+Cd*Cb*Db)*Rh^6
Estrab1=(12*R1+4*R1^3+2*C1*Ca*Cb+2*D1*Da*Db+Ac*Cb+Ad*Db+Bc*Ca+Bd*Da+Cd*Ca*
Db)*Rh^5
```



```
2090  Epota1=(-(1/3+1/5*Ca+1/6*Da)*R1+(1/3*Ca+5/12*Da)*R1^3)*Rh^4
2100  Epotb1=(-(1/4+1/5*Cb+1/6*Db)*R1+(1/3*Cb+5/12*Db)*R1^3)*Rh^5
2110  !HERE  FOLLOW  CALCULATIONS  OF THE COEFFICIENTS IN THE ENERGY EXPRESSION
FOR LAYER 2
2120  Q1=16/3*R+4/3*R^3
2130  Q2=36/5*R+4*R^3
2140  Q3=64/7*R+44/5*R^3+24/5*R^5+4/7*R^7
2150  Q4=100/9*R+100/7*R^3+12*R^5+100/21*R^7
2160  Q5=4*R^3+36/5*R^5
2170  Q6=8*(R-R^3)
2180  Q7=8*(R-5/3*R^3)
2190  Q8=12*R+4*R^3
2200  Q9=64/5*R-8/3*R^3-8/5*R^5
2210  Q10=40/3*R-10*R^3-4*R^5
2220  Q11=16*R+6*R^3-12/5*R^5
2230  Q12=120/7*R-8*R^5
2240  Q13=20*(R+R^3+3/5*R^5+1/7*R^7)
2250  Q14=-10*R^3-12*R^5
2260  Q15=-24*R^3+24/5*R^5
2270  Q16=-16*(R^3+R^5)
2280  Q17=-30*R^3+12*R^5
2290  !Estra2,Estrb2= AND Estrab2 ARE FACTORS TO MULTIPLY WITH A^2*H2^4,B^2*H2^6
 AND A*B*H2^5 IN THE STRAIN ENERGY EXPRESSION FOR LAYER 2
2300  !!!
2310  Estra12=4*R*A22a^2+Q1*A23a^2+Q2*A24a^2+Q3*A25a^2+Q4*A26a^2+12*R^3*A41a^2+Q
5*A42a^2+A22a*B*(A23a*R+A24a*R+Q6/8*A25a+Q7/8*A26a+A42a*R^3)
2320  Estra22=A23a*(Q8*A24a+Q9*A25a+Q10*A26a-8*R^3*A41a+4*R^3*A42a)+A24a*(Q11*A2
5a+Q12*A26a-12*R^3*A41a)+A25a*(Q13*A26a+Q14*A42a+Q15*A41a)+12*R^3*A41a*A42a
2330  Estra2=Estra12+Estra22+A26a*(Q16*A42a+Q17*A41a)
2340  !!!
2350  Estrb12=4*R*A22b^2+Q1*A23b^2+Q2*A24b^2+Q3*A25b^2+Q4*A26b^2+12*R^3*A41b^2+Q
5*A42b^2+A22b*B*(A23b*R+A24b*R+Q6/8*A25b+Q7/8*A26b+A42b*R^3)
2360  Estrb22=A23b*(Q8*A24b+Q9*A25b+Q10*A26b-8*R^3*A41b+4*R^3*A42b)+A24b*(Q11*A2
5b+Q12*A26b-12*R^3*A41b)+A25b*(Q13*A26b+Q14*A42b+Q15*A41b)+12*R^3*A41b*A42b
2370  Estrb2=Estrb12+Estrb22+A26b*(Q16*A42b+Q17*A41b)
2380  !!!
2390  Estrab12=8*R*A22a*A22b+2*Q1*A23a*A23b+2*Q2*A24a*A24b+2*Q3*A25a*A25b+2*Q4*A
26a*A26b+24*R^3*A41a*A41b+2*Q5*A42a*A42b+A22a*(8*A23b*R+8*A24b*R+Q6*A25b)
2400  Estrab22=A22a*(Q7*A26b+8*R^3*A42b)+A22b*8*(A23a*R+A24a*R+Q6/8*A25a+Q7/8*A2
6a+R^3*A42a)+A23a*(Q8*A24b+Q9*A25b+Q10*A26b-8*R^3*A41b+4*R^3*A42b)
2410  Estrab32=A23b*(Q8*A24a+Q9*A25a+Q10*A26a-8*R^3*A41a+4*R^3*A42a)+A24a*(Q11*A
25b+Q12*A26b-12*R^3*A41b)+A24b*(Q11*A25a+Q12*A26a-12*R^3*A41a)
2420  Estrab42=A25a*(Q13*A26b+Q14*A42b+Q15*A41b)+A25b*(Q13*A26a+Q14*A42a+Q15*A41
a)+A26a*(Q16*A42b+Q17*A41b)+A26b*(Q16*A42a+Q17*A41a)+12*R^3*A41a*A42b
2430  Estrab2=Estrab12+Estrab22+Estrab32+Estrab42+12*R^3*A41b*A42a
2440  !!!
2450  Estra=Mu1*Estra1+Mu2*Estra2
2460  Estrb=Mu1*Estrb1+Mu2*Estrb2
2470  Estrab=Mu1*Estrab1+Mu2*Estrab2
2480  !!!
2490  !BELOW FOLLOW CALCULATION OF THE FACTORS TO MULTIPLY WITH A*H2^4 AND B*H2^
5 IN THE EXPRESSION FOR POTENTIAL ENERGY IN LAYER 2
2500  Epota2=(A21a+.5*A22a+1/3*A23a+1/4*A24a+1/5*A25a+1/6*A26a)*R+(A41a+1/2*A42a
-1/3*A25a-5/12*A26a)*R^3
2510  Epotb2=(A21b+.5*A22b+1/3*A23b+1/4*A24b+1/5*A25b+1/6*A26b)*R+(A41b+1/2*A42b
-1/3*A25b-5/12*A26b)*R^3
2520  !!!
2530  Epota=Ro2*Gr*(Epota1*Rro+Epota2)
2540  Epotb=Ro2*Gr*(Epotb1*Rro+Epotb2)
2550  SUBEND
```


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CRACK-SEAL MECHANISM IN A LIMESTONE: A FACTOR OF DEFORMATION IN STRIKE-SLIP FAULTING

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ABSTRACT

Gaviglio, P., 1986. Crack-seal mechanism in a limestone: a factor of deformation in strike-slip *Tectonophysics*, 131: 247–255.

A repeated crack-seal mechanism may play a major part in the deformation process in a sl This paper develops a microscopic approach to the association of shear planes and tensi between two en échelon faults. The simultaneous development of these fractures yields a ver pattern.

The structure, discovered in a Cretaceous limestone of the Arc basin (Bouches du Rhône, l located between two left-stepping sinistral strike-slip faults. The horizontal displacement does n a few centimeters: therefore the structure may be representative of the early stage of deformat a brittle shear zone. Each strain increment includes both shearing and tensile fracturing. Inhor conditions of friction seem to be prevailing.

INTRODUCTION

Faults are discontinuous fractures at all length scales (Tchalenko, 1970; al., 1976; Segall and Pollard, 1980). The existence of tension cracks is func to the early stages of fault development (Beach, 1975), as well as throug deformation of the shear zone (Tchalenko and Ambraseys, 1970; Rayn Delair, 1978; Segall and Pollard, 1980; Gamond, 1983).

The crack-seal mechanism is a significant process of natural rock def (Ramsay, 1980): a tension crack forms under accumulation of elastic strai ated with fluid pressure and is filled by crystallized minerals through transfer; repetition of cracking and sealing forms compound veins c fibrous crystals and, possibly, inclusion bands parallel to the rock wall. C increments form according to various processes of fibre growth (Cox and F 1983). Most of the described tension veins displaying evidence of thi incremental strain seem to have resulted from simple shortening and stret

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